Direct Stiffness Method – Constraints

Simulation Methods in Acoustics

Single Degree of Freedom Constraints – SDFC

Unconstrained system

$$\mathbf{K}\mathbf{u} = \mathbf{f} \tag{1}$$

Some Nodal displacement values are prescribed:

$$\mathbf{u}_c = \bar{\mathbf{u}}(= \mathbf{0}) \tag{2}$$

SDFC: Each constraint equation addresses 1 DOF
Partitioning: c: constrained, u: unconstrained

$$\begin{bmatrix} \mathbf{K}_{cc} & \mathbf{K}_{cu} \\ \mathbf{K}_{uc} & \mathbf{K}_{uu} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_c = \bar{\mathbf{u}} \\ \mathbf{u}_u \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_c \\ \mathbf{f}_u \end{Bmatrix} \tag{3}$$

Reduced system for the unconstrained displacements:

$$\mathbf{K}_{uu}\mathbf{u}_u = \mathbf{f}_u - \mathbf{K}_{uc}\bar{\mathbf{u}} \tag{4}$$

Multiple Degree of Freedom Constraints

Examples:

	canonical form
$u_1 = u_3$	$u_1 - u_3 = 0$
$u_1 = \frac{u_2 + u_3}{2}$	$u_1 - \frac{1}{2}u_2 - \frac{1}{2}u_3 = 0$

General canonical form:

$$\sum_i a_i u_i = 0 \tag{5}$$

Multiple (K) constraint equations for N DOF

$$\mathbf{Au} = \mathbf{0}, \qquad \mathbf{A} \in \mathbb{R}^{K \times N} \tag{6}$$

Objective: Solve $\mathbf{Ku} = \mathbf{f}$ so that $\mathbf{Au} = \mathbf{0}$ holds.

MDFC – Master Slave Method

- Assume that $\mathbf{A}$ contains K *independent* constraints: $\mathrm{rank}\mathbf{A} = K$
- Each constraint lowers the # unknowns by one
- $N - K$ master DOF
- K slave DOF are completely determined by the master DOF

Partition the DOF so that $\mathbf{A}_s$ is invertible

$$\begin{bmatrix} \mathbf{A}_m & \mathbf{A}_s \end{bmatrix} \begin{Bmatrix} \mathbf{u}_m \\ \mathbf{u}_s \end{Bmatrix} = \mathbf{0} \qquad \rightarrow \qquad \mathbf{u}_s = -\mathbf{A}_s^{-1}\mathbf{A}_m\mathbf{u}_m \tag{7}$$

Total - Master dependence

$$\mathbf{u} = \begin{Bmatrix} \mathbf{u}_m \\ \mathbf{u}_s \end{Bmatrix} = \begin{bmatrix} \mathbf{I} \\ -\mathbf{A}_s^{-1}\mathbf{A}_m \end{bmatrix} \mathbf{u}_m = \mathbf{T}\mathbf{u}_m \tag{8}$$

$$\mathbf{K}\mathbf{T}\mathbf{u}_m = \mathbf{f} \tag{9}$$

Final constrained $N - K \times N - K$ system:

$$\mathbf{T}^{\mathrm{T}}\mathbf{K}\mathbf{T}\mathbf{u}_m = \mathbf{T}^{\mathrm{T}}\mathbf{f}, \qquad \hat{\mathbf{K}}\mathbf{u}_m = \hat{\mathbf{f}} \tag{10}$$

Master-Slave Method – Simple connections

Unconnected 1D system of two springs

$$\begin{bmatrix} K_1 & -K_1 & & \\ -K_1 & K_1 & & \\ & & K_2 & -K_2 \\ & & -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{Bmatrix} \tag{11}$$

Constraint: $u_2 = u_3$. Slave: u_3, Masters: u_1, u_2, u_4.

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ & 1 & \\ & & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_4 \end{Bmatrix} \tag{12}$$

$$\mathbf{T}^{\mathrm{T}}\mathbf{K}\mathbf{T} = \begin{bmatrix} K_1 & -K_1 & \\ -K_1 & K_1 + K_2 & -K_2 \\ & -K_2 & K_2 \end{bmatrix}, \qquad \mathbf{T}^{\mathrm{T}}\mathbf{f} = \begin{Bmatrix} f_1 \\ f_2 + f_3 \\ f_4 \end{Bmatrix} \tag{13}$$

Master Slave Method – Rigid body motion

- Slave nodes connected by a massless rigid body
- Master DOF: Displacements u_{x0}, u_{y0} and rotation θ_0 around (x_0, y_0):
- Displacement field of the rigid body

$$u_x(x, y) = u_{x0} - (y - y_0)\theta_0 \tag{14}$$
$$u_y(x, y) = u_{y0} + (x - x_0)\theta_0 \tag{15}$$

- Master-slave constraints for one slave node:

$$\mathbf{u}_s = \begin{bmatrix} 1 & & -(y - y_0) \\ & 1 & x - x_0 \end{bmatrix} \begin{Bmatrix} u_{x0} \\ u_{y0} \\ \theta_0 \end{Bmatrix} \tag{16}$$

MDFC – The Penalty Method

$$\mathbf{K}\mathbf{u} = \mathbf{f}, \qquad \mathbf{A}\mathbf{u} = \mathbf{0} \tag{17}$$

Example constraint: $u_i = u_j \rightarrow u_i$ and u_j are rigidly connected

$$\begin{bmatrix} 1 & -1 \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = 0 \tag{18}$$

Represent by a ,,very stiff" spring:

$$w \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \qquad \mathbf{K}^p\mathbf{u} = \mathbf{0} \tag{19}$$

Solve

$$\left(\mathbf{K} + \mathbf{K}^p\right)\mathbf{u} = \mathbf{f}, \qquad w \rightarrow \infty \tag{20}$$

Square rule: $w = \max K \cdot 10^{d/2}$, where the system precision is d decimal digits

MDFC – The Penalty Method

General $k-th$ constraint: $\mathbf{a}_k\mathbf{u} = 0$
Create an element by premultiplying by $\mathbf{a}_k^{\mathrm{T}}$

$$\underbrace{w\mathbf{a}_k^{\mathrm{T}}\mathbf{a}_k}_{\mathbf{K}_k^p}\mathbf{u} = \mathbf{0} \tag{21}$$

Final system of equations

$$\left(\mathbf{K} + \sum_k \mathbf{K}_k^p\right)\mathbf{u} = \mathbf{f} \tag{22}$$

or equivalently

$$\left(\mathbf{K} + w\mathbf{A}^{\mathrm{T}}\mathbf{A}\right)\mathbf{u} = \mathbf{f} \tag{23}$$

MDFC – The Lagrange Method

The K constraints are taken into account as an additional force term defined by K Lagrange multipliers $\boldsymbol{\lambda}$

$$\mathbf{K}\mathbf{u} = \mathbf{f} - \mathbf{A}^{\mathrm{T}}\boldsymbol{\lambda} \tag{24}$$

The total system to solve takes the form

$$\begin{bmatrix} \mathbf{K} & \mathbf{A}^{\mathrm{T}} \\ \mathbf{A} & \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ \lambda \end{Bmatrix} = \begin{Bmatrix} \mathbf{f} \\ \mathbf{0} \end{Bmatrix}$$

Advantages

- Symmetric and sparse
- Lagrange multipliers obtained from the solution

Disadvantage:

- Bandwidth increases