Time stepping methods

Simulation Methods in Acoustics

Motivation

- For a dynamical system, solve the equation of motion:

$$\mathbf{K}\mathbf{u}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{M}\ddot{\mathbf{u}}(t) = \mathbf{f}(t)$$

  We seek $\mathbf{u}(t)$ for a known excitation $\mathbf{f}(t)$.
- This can be done using the impulse response, and matrix exponentials or computing a spectral decomposition (This gives $\mathbf{u}(t)$ analytically, as a continuous function of t)
- If the system is large, these methods can be too involving
- Instead of a continuous function, we compute an *approximation at discrete times* ($t_k = t_0 + kh$, time step: h)
- These approximation methods are referred to as *time stepping methods* in general

The forward Euler method

- Time stepping methods are constructed from Taylor series

$$x(t_k + h) = x(t_k) + h\dot{x}(t_k) + \frac{h^2}{2}\ddot{x}(t_k) + \ldots + \frac{h^n}{n!}x^{(n)}(t_k)$$

  The equality is exact, as $n \to \infty$

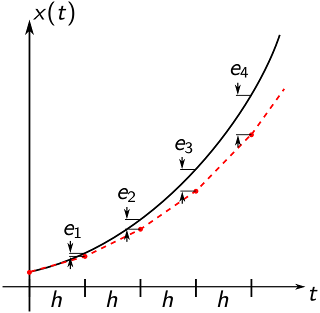
- Approximation by *truncation* of the Taylor series

$$x(t_k + h) = x(t_k) + h\dot{x}(t_k) + O(h^2)$$
$$x(t_k + h) \approx x(t_k) + h\dot{x}(t_k) \qquad \text{with local error } O(h^2)$$

- Truncating at first order, we get the *forward Euler method*

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b} \qquad \text{(example eq.)}$$
$$\mathbf{x}_{k+1} = \mathbf{x}_k + h(\mathbf{A}\mathbf{x} + \mathbf{b})$$

Accumulation of the approximation error



- Solve $\dot{(x)} = \lambda x$, $x(0) = 1$ (solution is: $x(t) = \mathrm{e}^{\lambda t}$) using the forward Euler method. We get the approximation in red.
- As the derivative is approximated in each point, the error of the solution accumulates!
- The number of time steps is $O(1/h)$, in each time step we add an error $O(h^2)$, thus the total error is $O(h)$.
- Therefore, the forward Euler method is *first order accurate*.

Stability vs. Accuracy

- Time stepping scheme have two important properties: *stability region* and *order of accuracy*. It is very important to distinguish these properties.
- **Stability**: The stability region is the domain of parameters (time step h), in which it is *guaranteed* that the error of the approximation will not grow exponentially.[1] However, stability does not inform us on the magnitude of the error.
- **Accuracy**: The order of accuracy relates the choice of parameters to the order of the error of the time stepping scheme. However, accuracy does not provide any information on the stability region.

[1] Provided that the analytical solution is bounded.

Stability analysis of the forward Euler method I.

- Introduce the simple model equation:

$$\dot{x}(t) = \lambda x(t) \qquad x(0) = x_0$$

- This is useful for multidimensional systems as well, as λ can be any eigenvalue of the system. All eigenvalues should be in the stability region of the method.
- Apply the forward Euler method throughout n steps

$$x_n = (1 + h\lambda)^n x_0$$

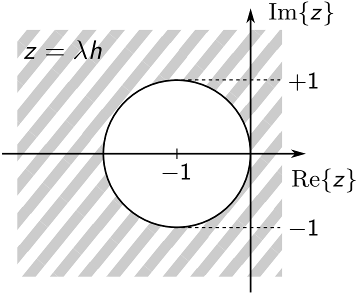
- Stability achieved if $|1 + h\lambda| \leq 1$

Stability analysis of the forward Euler method II.

- Rearrange: $|1 + h\lambda| \leq 1 \rightarrow -1 \geq 1 + \lambda h \leq 1$

$$0 \leq h \leq \frac{2}{|\lambda|}$$

- Even if original equation is stable, forward Euler scheme is only *conditionally stable*. Stability region shown in the figure.
- The condition means an upper limit on the time step h
- Remember, this tells *nothing* about the accuracy.



The backward Euler method

- Write the Taylor series trunction as

$$
\begin{aligned}
x(t_k - h) &= x(t_k) - h\dot{x}(t_k) && + O(h^2) \\
x(t_k) &= x(t_k + h) - h\dot{x}(t_k + h) && + O(h^2)
\end{aligned}
$$

- Use it for the system: $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}$

$$
\begin{aligned}
\mathbf{x}_k &= \mathbf{x}_{k+1} - h\left(\mathbf{A}\mathbf{x}_{k+1} + \mathbf{b}_{k+1}\right) \\
\left(\mathbf{I} - h\mathbf{A}\right)\mathbf{x}_{k+1} &= \mathbf{x}_k + h\mathbf{b}_{k+1}
\end{aligned}
$$

- This is an *implicit method*, as $\mathbf{x}_{k+1}$ appears on both sides of the equation.
- By a similar argument as above, the backward Euler method is also $O(h)$ accurate.
- In general, implicit time steps require the solution of a system of equations, as in this case, while explicit methods (such as forward Euler) require only matrix–vector products.

Stability of the backward Euler method

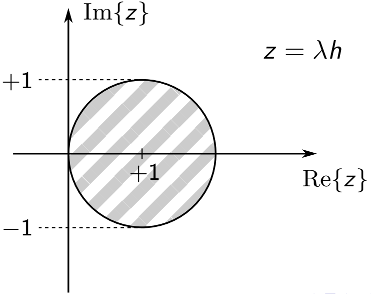
- As before, use model equation: $\dot{x}(t) = \lambda x(t)$ for n steps

$$x_n = \left(\frac{1}{1 - h\lambda} \right)^n x_0$$

- We get the stability region as:

$$\left| \frac{1}{1 - h\lambda} \right| \leq 1 \quad \rightarrow \quad \frac{1}{|1 - h\lambda|} \leq 1 \quad \rightarrow \quad 1 \leq |1 - h\lambda|$$

- If $\lambda < 0$, then we get $1 \leq 1 + h|\lambda|$, which means that the method is *unconditionally stable*, for any time step size h



The central derivative

- Construct the following quantities by Taylor series

$$f(t+h) = f(t) + hf'(t) + \frac{h^2}{2}f''(t) + O(h^3)$$
$$f(t-h) = f(t) - hf'(t) + \frac{h^2}{2}f''(t) + O(h^3)$$

- Subtract the second equation from the first to get:

$$f'(t) = \frac{f(t+h) - f(t-h)}{2h} + O(h^3)$$

- The central derivative is $O(h^3)$ accurate (this is local accuracy)

The trapezoid method

- Use the central derivative to construct a second order accurate implicit scheme

$$x_{k+1} = x_k + h\dot{x}_{k+1/2} + O(h^3) \qquad \dot{x}_{k+1/2} = \frac{\dot{x}_{k+1} + \dot{x}_k}{2} + O(h^2)$$

- Using $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}$

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \frac{h}{2}\left[\mathbf{A}\mathbf{x}_{k+1} + \mathbf{b}_{k+1} + \mathbf{A}\mathbf{x}_k + \mathbf{b}_k\right]$$

- We get the implicit form as

$$\left(\mathbf{I} - \frac{h}{2}\mathbf{A}\right)\mathbf{x}_{k+1} = \mathbf{x}_k + \frac{h}{2}\left[\mathbf{A}\mathbf{x}_k + \mathbf{b}_{k+1} + \mathbf{b}_k\right]$$

- The resulting *trapezoid method* is $O(h^2)$ accurate (global accuracy) and unconditionally stable.

Runge – Kutta methods I.

- Let's write the quantity x_{k+1} using the integral of $\dot{x}$

$$x_{k+1} = x_k + \int_{t_k}^{t_k+h} \dot{x}(t)\,\mathrm{d}t$$

- Runge – Kutta methods approximate the integral by means of simple quadratures (brick, trapezoid, Simpson's rule etc.)
- In general: a *quadrature* method is the approximation of the integral by a weighted sum of function values:

$$\int_{t_k}^{t_k+h} f(t)\,\mathrm{d}t \approx \sum_{i=1}^{n} w_i f(t_i)$$

With base points t_i and weights w_i.

Runge–Kutta methods II.

- Write the general form $\dot{x} = F(x)$ for simplicity
- We can express $F(x_{k+1})$ using the forward Euler as

$$F(x_{k+1}) = F[\underbrace{x_k + hF(x_k) + O(h^2)]}_{\text{Forward Euler}} = \underbrace{F[x_k + hF(x_k)] + O(h^2)}_{\text{Taylor series}}$$

- Apply the above to approximate x_{k+1} as

$$x_{k+1} = x_k + \frac{h}{2}\left[F(x_k) + F(x_k + hF(x_k))\right] + O(h^3)$$

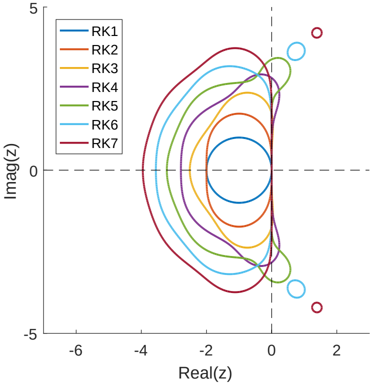
  This is *Heun's method*, a 2nd order accurate explicit scheme
- The RK4 form is also applied often in practice:

$$x_{k+1} = x_k + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad \begin{aligned} k_1 &= F(x_k) \\ k_2 &= F(x_k + \tfrac{1}{2}hk_1) \\ k_3 &= F(x_k + \tfrac{1}{2}hk_2) \\ k_4 &= F(x_k + hk_3) \end{aligned}$$

- Similarly, other R–K forms can be constructed

Stability of Runge–Kutta methods

- Runge–Kutta methods are attractive because they have
  1. Low computational cost (because of their explicitness)
  2. High accuracy, e.g. RK4 is $O(h^4)$
  3. Extended stability, compared to the forward Euler method
- Figure below shows the stability regions (RK1 $\equiv$ FW Euler)



A simple example

- Let's consider a simple damped oscillator (mass, spring, damper) with a single degree of freedom

$$m = 1\,\mathrm{kg} \qquad c = 0.5\,\mathrm{Ns/m} \qquad k = 10\,\mathrm{N/m}$$

- We can write the system of equations as

$$\begin{Bmatrix} \ddot{u} \\ \dot{u} \end{Bmatrix} = \begin{bmatrix} -m/c & -m/k \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} \dot{u} \\ u \end{Bmatrix} + \begin{Bmatrix} f/m \\ 0 \end{Bmatrix}$$

- Or in matrix form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}$$

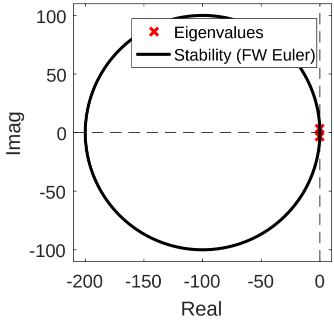
- Solve with zero r.h.s. using the initial condition:

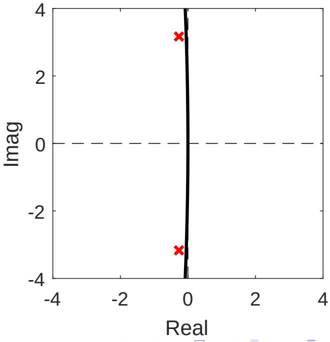
$$\mathbf{x}(0) = \begin{Bmatrix} v(0) \\ x(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

- Use different time stepping methods to observe their typical behavior (use $h = 0.01\,\mathrm{s}$)

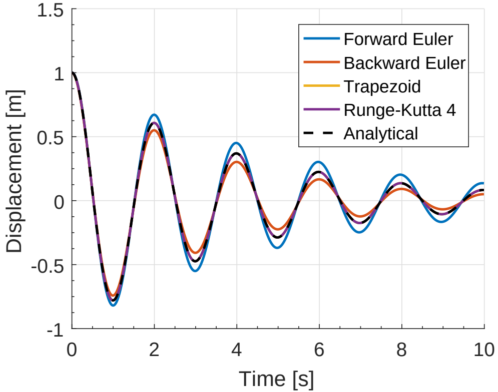
Behavior of the FW Euler method

- The stability region of the forward Euler method is clearly a limiting factor, when we have oscillating solutions with small damping (λ near the imaginary axis)
- In these cases, we need extremely small time step sizes to get a stabile solution
- The configuration for the above example is shown in the figure





Results



- Observations:
  - Forward Euler underestimates the damping (generally true)
  - Backward Euler overestimates the damping (generally true)
  - The differences in orders of accuracy are clearly visible (the trapezoid and RK4 are very close to the analytical)