The modal solution

Simulation Methods in Acoustics

December 13, 2017

Solution in Frequency domain

Equation of motion of a dynamic system in time domain

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{f}(t)$$

Equation of motion in frequency domain (Fourier transformed system)

$$\begin{aligned} \mathbf{u}(\omega) &= \mathcal{F}\left\{\mathbf{u}(t); \omega\right\} \\ \mathbf{f}(\omega) &= \mathcal{F}\left\{\mathbf{f}(t); \omega\right\} \\ \underbrace{\left[\mathbf{K} + \mathrm{j}\omega\mathbf{C} - \omega^2\mathbf{M}\right]}_{\text{dynamic stiffness:}\mathbf{S}(\omega)} \mathbf{u}(\omega) &= \mathbf{f}(\omega) \end{aligned}$$

Dynamic stiffness:

$$\begin{aligned} \mathbf{S}(\omega) &= \mathbf{K} + \mathrm{j}\omega\mathbf{C} - \omega^2\mathbf{M} \\ \lim_{\omega\to 0} \mathbf{S}(\omega) &= \mathbf{K} \\ \lim_{\omega\to\infty} \mathbf{S}(\omega) &= -\omega^2\mathbf{M} \end{aligned}$$

Normal modes

Modes: nontrivial solutions of the undamped system without excitation
Mode shapes ψ_j and eigenfrequencies ω_j are solutions of the generalised eigenvalue problem

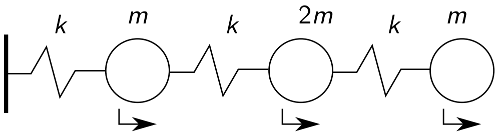
$$\mathbf{K}\psi_j = \omega_j^2 \mathbf{M}\psi_j, \qquad j = 1 \dots N$$

- $\mathbf{K}$ symm, positive semidefinite. $\mathbf{M}$ symm, positive definite.
- nonnnegative real eigenvalues $\omega^2 \geq 0$.
- modes are orthogonal to the mass matrix
- mode amplitudes can be chosen as mass-orthonormal

$$\psi_i^{\mathrm{T}} \mathbf{M} \psi_j = \delta_{ij} \quad \mathbf{\Psi}^{\mathrm{T}} \mathbf{M} \mathbf{\Psi} = \mathbf{I}$$

$$\psi_i^{\mathrm{T}} \mathbf{K} \psi_j = \delta_{ij}\omega_i^2 \quad \mathbf{\Psi}^{\mathrm{T}} \mathbf{K} \mathbf{\Psi} = \mathbf{\Lambda} = \mathrm{diag}\left\{\omega_i^2\right\}$$

Example



$$\mathbf{K} = k \begin{bmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 1 \end{bmatrix}, \quad \mathbf{M} = m \cdot \mathrm{diag}\{1, 2, 1\}$$

$$\omega_j = \sqrt{\frac{k}{m}} \cdot \begin{Bmatrix} 0.3813 \\ 1.1845 \\ 1.5658 \end{Bmatrix} \qquad \boldsymbol{\Psi} = \frac{1}{\sqrt{m}} \left[\begin{Bmatrix} 0.28 \\ 0.52 \\ 0.61 \end{Bmatrix} \quad \begin{Bmatrix} -0.50 \\ -0.30 \\ 0.75 \end{Bmatrix} \quad \begin{Bmatrix} -0.81 \\ 0.37 \\ -0.26 \end{Bmatrix} \right]$$

The Modal Solution

Frequency domain undamped system with excitation

$$\left(\mathbf{K} - \omega^2 \mathbf{M}\right) \mathbf{u}(\omega) = \mathbf{f}(\omega)$$

Modal superposition: displacement in modal base

$$\mathbf{u}(\omega) = \sum_{j=0}^{N} \psi_j q_j(\omega) = \mathbf{\Psi} \mathbf{q}(\omega)$$

Substitution into the system and premultiplication by $\mathbf{\Psi}^{\mathrm{T}}$

$$\mathbf{\Psi}^{\mathrm{T}} \left(\mathbf{K} - \omega^2 \mathbf{M}\right) \mathbf{\Psi} \mathbf{q}(\omega) = \mathbf{\Psi}^{\mathrm{T}} \mathbf{f}(\omega)$$

$$\left(\mathbf{\Lambda} - \omega^2 \mathbf{I}\right) \mathbf{q}(\omega) = \mathbf{\Psi}^{\mathrm{T}} \mathbf{f}(\omega)$$

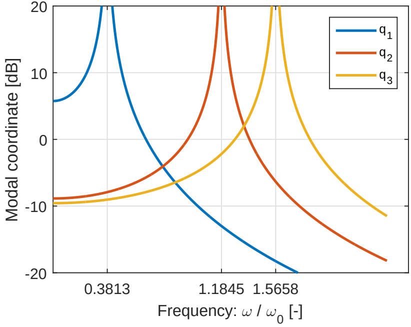
The lhs system matrix is diagonal: The modal coordinates are uncoupled

$$q_i(\omega) = \frac{\psi_i^{\mathrm{T}} \mathbf{f}(\omega)}{\omega_i^2 - \omega^2}, \quad \mathbf{u}(\omega) = \sum_{j=1}^{N} \frac{\psi_j \psi_j^{\mathrm{T}} \mathbf{f}(\omega)}{\omega_j^2 - \omega^2}$$

Example – continued

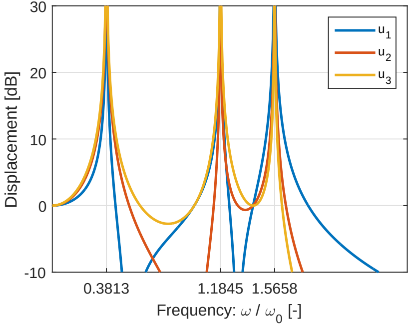
- Excitation: unit force f_1

Modal coordinates



$$\omega_0 = \sqrt{k/m}$$

Displacement response



Modal Solution: Benefits

- The modes are uncoupled. The modal coordinate q_i can be computed from the excitation, without any information on other modes
- System size:
  The modal superposition can be truncated at $D \ll N$ modes, if the Rubin criterion is fulfilled.

  $$\omega_D > 1.5\omega_{\max}$$

  where $\omega_{\max}$ is the upper frequency limit of investigations

Damped Modal Systems – Proportional Damping

Modal superposition with general damping **C**

$$\left(\boldsymbol{\Lambda} + \mathrm{j}\omega \underbrace{\boldsymbol{\Psi}^{\mathrm{T}}\mathbf{C}\boldsymbol{\Psi}}_{\text{not diagonal}} - \omega^2\mathbf{I}\right)\mathbf{q}(\omega) = \boldsymbol{\Psi}^{\mathrm{T}}\mathbf{f}(\omega)$$

Modal coordinates are coupled by a general damping matrix

Approximate solution: Proportional damping models

$$\mathbf{C} \approx \alpha\mathbf{K} + \beta\mathbf{M}$$

The resulting diagonal system

$$\left(\boldsymbol{\Lambda} + \mathrm{j}\omega\left(\alpha\boldsymbol{\Lambda} + \beta\mathbf{I}\right) - \omega^2\mathbf{I}\right)\mathbf{q}(\omega) = \boldsymbol{\Psi}^{\mathrm{T}}\mathbf{f}(\omega)$$

Resulting modal damping coefficients:

$$\eta_i = \alpha\omega_i + \frac{\beta}{\omega_i}$$

α and β typically fitted to experiments

Modal damping coefficients

Modal damping coefficients can be determined from state space like eigenvalue problem

$$\begin{bmatrix} \mathbf{K} & \mathbf{C} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} = \mathrm{j}\omega \begin{bmatrix} \mathbf{0} & -\mathbf{M} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix}$$

The eigenvalues ω_i are complex, and the modal damping is determined as $\eta_i = \frac{\Im \omega_i}{\Re \omega_i}$