The Finite Element Method – Basics

Simulation Methods in Acoustics

Acoustic Wave Propagation in Closed Volumes – BVP

Homogeneous Helmholtz equation

$$\nabla^2 p(\mathbf{x}) + k^2 p(\mathbf{x}) = 0, \qquad \mathbf{x} \in \Omega$$

Euler equation

$$\nabla p(\mathbf{x}) + \mathrm{i}\omega\rho_0 \mathbf{v}(\mathbf{x}) = \mathbf{0}, \qquad \mathbf{x} \in \Omega$$

Dirichlet BC:

$$p(\mathbf{x}) = \bar{p}(\mathbf{x}), \qquad \mathbf{x} \in \Gamma$$

Neumann BC:

$$\frac{\partial p(\mathbf{x})}{\partial n} = \nabla p(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) = -\mathrm{i}\omega\rho_0 \bar{v}_n(\mathbf{x}), \qquad \mathbf{x} \in \Gamma$$

Robin BC:

$$p(\mathbf{x}) - \bar{z}(\mathbf{x}) v_n(\mathbf{x}) = 0, \qquad \mathbf{x} \in \Gamma$$

Acoustic Wave Propagation in 1D Closed Volumes – BVP

Homogeneous Helmholtz equation

$$p''(x) + k^2 p(x) = 0, \qquad 0 \leq x \leq L$$

Euler equation

$$p'(x) + \mathrm{i}\omega\rho_0 v(x) = 0, \qquad 0 \leq x \leq L$$

Dirichlet BC:

$$p(x) = \bar{p}(x), \qquad x \in \{0, L\}$$

Neumann BC:

$$p'(x) = -\mathrm{i}\omega\rho_0 \bar{v}(x), \qquad x \in \{0, L\}$$

Robin BC:

$$p(x) - \bar{z}(x) v(x) = 0, \qquad x \in \{0, L\}$$

Weak form in 1D

Formulate the weak form of the BVP:

$$\int \psi(x)p''(x)\mathrm{d}x + k^2 \int \psi(x)p(x)\mathrm{d}x = 0$$

Integration by parts

$$\left[\psi(x)p'(x)\right]_0^L - \int \psi'(x)p'(x)\mathrm{d}x + k^2 \int \psi(x)p(x)\mathrm{d}x = 0$$

Boundary condition and rearranging

$$\rho_0 c^2 \int \psi'(x)p'(x)\mathrm{d}x - \omega^2 \rho_0 \int \psi(x)p(x)\mathrm{d}x = -\mathrm{i}\omega z_0^2 \left[\psi(x)v(x)\right]_0^L$$

Finite Dimensional Approximation

Finite dimension approximation with N shape functions $N_j(x)$

$$p(x) = \sum_{j=1}^{N} N_j(x) p_j = \begin{bmatrix} N_1(x) & \dots & N_n(x) \end{bmatrix} \begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} = \mathbf{N}(x)\mathbf{p}$$

Galerkin method: same base for the test function $\psi(x)$

$$\psi(x) = \sum_{i=1}^{N} N_i(x)\psi_i = \mathbf{N}(x)\boldsymbol{\psi}$$

$$\rho_0 c^2 \cancel{\boldsymbol{\psi}^{\mathrm{T}}} \int_\Omega \mathbf{N}'(x)^{\mathrm{T}} \mathbf{N}'(x) \mathrm{d}x \mathbf{p} - \omega^2 \rho_0 \cancel{\boldsymbol{\psi}^{\mathrm{T}}} \int_\Omega \mathbf{N}(x)^{\mathrm{T}} \mathbf{N}(x) \mathrm{d}x \mathbf{p} = -\mathrm{i}\omega z_0^2 \cancel{\boldsymbol{\psi}^{\mathrm{T}}} \left[\mathbf{N}(x)^{\mathrm{T}} v(x)\right]_0^L$$

System matrices

The resulting system in matrix form

$$\left(\mathbf{K} - \omega^2 \mathbf{M}\right) \mathbf{p} = -\mathrm{i}\omega \mathbf{q}$$

Where the matrices and the load vector are defined as

$$\mathbf{K} = \rho_0 c^2 \int_\Omega \mathbf{N}'(x)^\mathrm{T} \mathbf{N}'(x) \mathrm{d}x$$

$$\mathbf{M} = \rho_0 \int_\Omega \mathbf{N}(x)^\mathrm{T} \mathbf{N}(x) \mathrm{d}x$$

$$\mathbf{q} = z_0^2 \left[\mathbf{N}(x)^\mathrm{T} v(x)\right]_0^L$$

Elements

Integrals computed element by element

$$\mathbf{M} = \rho_0 \int_\Omega \mathbf{N}(x)^{\mathrm{T}} \mathbf{N}(x) \mathrm{d}x = \sum_e \underbrace{\rho_0 \int_{\Omega_e} \mathbf{N}(x)^{\mathrm{T}} \mathbf{N}(x) \mathrm{d}x}_{\mathbf{M}_e}$$

Integration performed in reference domain

$$\mathbf{M}_e = \rho_0 \int_{-1}^{+1} \mathbf{N}(x^e(\xi))^{\mathrm{T}} \mathbf{N}(x^e(\xi)) \frac{\mathrm{d}x^e(\xi)}{\mathrm{d}\xi} \mathrm{d}\xi$$

Where the mapping between the reference domain and the element domain is

$$x^e(\xi) = \sum_{k=1}^{K} x_k^e L_k(\xi), \qquad \xi \in \Xi$$

Which, in our simple 1D case is

$$x^e(\xi) = x_1^e \frac{1-\xi}{2} + x_2^e \frac{1+\xi}{2}, \qquad -1 \le \xi \le +1$$

Mass matrix

The Jacobian of the coordinate transform is

$$J_e(\xi) = \frac{\mathrm{d}x^e(\xi)}{\mathrm{d}\xi} = \sum_{k=1}^{K} x_k^e L'_k(\xi) = \frac{x_2^e - x_1^e}{2}$$

Pressure and testing shape functions are usually already defined in the reference domain: $N(x^e(\xi)) = N(\xi)$

$$\begin{aligned}\mathbf{M}_e &= \rho_0 \int_{-1}^{+1} \mathbf{N}(\xi)^{\mathrm{T}} \mathbf{N}(\xi) \frac{\mathrm{d}x^e(\xi)}{\mathrm{d}\xi} \mathrm{d}\xi \\ &= \rho_0 \frac{h_e}{2} \int_{-1}^{+1} \begin{bmatrix} \frac{1-\xi}{2} \\ \frac{1+\xi}{2} \end{bmatrix} \begin{bmatrix} \frac{1-\xi}{2} & \frac{1+\xi}{2} \end{bmatrix} \mathrm{d}\xi = \rho_0 \frac{h_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\end{aligned}$$

Stiffness matrix

$$\mathbf{K} = \rho_0 c^2 \int_\Omega \mathbf{N}'(x)^{\mathrm{T}} \mathbf{N}'(x) \mathrm{d}x = \sum_e \underbrace{\rho_0 c^2 \int_{\Omega_e} \mathbf{N}'(x)^{\mathrm{T}} \mathbf{N}'(x) \mathrm{d}x}_{\mathbf{K}_e}$$

Spatial derivatives also computed in the reference domain

$$\frac{\partial N(x)}{\partial x} = \frac{\frac{\partial N(x^e(\xi))}{\partial \xi}}{\frac{\partial x^e(\xi)}{\partial \xi}}$$

$$\begin{aligned}
\mathbf{K}_e &= \rho_0 c^2 \int_{-1}^{+1} \frac{\mathbf{N}'(\xi)^{\mathrm{T}}}{\frac{\mathrm{d}x^e(\xi)}{\mathrm{d}\xi}} \frac{\mathbf{N}'(\xi)}{\frac{\mathrm{d}x^e(\xi)}{\mathrm{d}\xi}} \frac{\mathrm{d}x^e(\xi)}{\mathrm{d}\xi} \mathrm{d}\xi \\
&= \rho_0 c^2 \frac{2}{h_e} \int_{-1}^{+1} \begin{bmatrix} \frac{-1}{2} \\ \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{-1}{2} & \frac{1}{2} \end{bmatrix} \mathrm{d}\xi = \frac{\rho_0 c^2}{h_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}
\end{aligned}$$

Solution with Neumann–Neumann BC

$$\left(\mathbf{K} - \omega^2 \mathbf{M}\right) \mathbf{p} = -\mathrm{i}\omega \mathbf{q}$$

$$\mathbf{K} = \frac{\rho_0 c^2}{h} \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & \ddots & \ddots & \ddots \\ & & & -1 & 1 \end{bmatrix},$$

$$\mathbf{M} = \rho_0 \frac{h}{6} \begin{bmatrix} 2 & 1 & & & \\ 1 & 4 & 1 & & \\ & 1 & 4 & 1 & \\ & & \ddots & \ddots & \ddots \\ & & & 1 & 2 \end{bmatrix}, \qquad \mathbf{q} = z_0^2 \begin{bmatrix} -\bar{v}(0) \\ 0 \\ 0 \\ \vdots \\ \bar{v}(L) \end{bmatrix}$$

Solution with Neumann–Robin BC

BC: $p(L) - z_2 v(L) = 0$
Substitution into the system results in a damping matrix term $\mathbf{C}$

$$\left(\mathbf{K} + \mathrm{i}\omega\mathbf{C} - \omega^2\mathbf{M}\right)\mathbf{p} = -\mathrm{i}\omega\mathbf{q}$$

and a modified load vector $\mathbf{q}$

$$\mathbf{C} = z_0^2 \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & \frac{1}{z_2} \end{bmatrix}, \qquad \mathbf{q} = z_0^2 \begin{bmatrix} -\bar{v}(0) \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$