Boundary Integral Equation Methods

Simulation Methods in Acoustics

The Method of Boundary Integral Equations

General steps of the procedure

1. Find Fundamental Solution of the PDE
2. Derive boundary integral representation formula (BIR)
   2.1 Testing (as for FEM)
   2.2 Integrating by parts, shift operator to testing function
   2.3 Apply boundary conditions
   2.4 Apply Fundamental Solution as testing function
3. Solve boundary integral equation (BIE)
   3.1 Discretisation of boundary
   3.2 Discretisation of fields on the boundary
   3.3 Galerkin / Collocation
4. Express solution in internal points by applying the BIR

Problem Statement

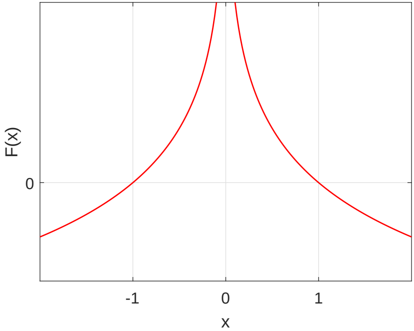
Static displacement of an ideal membrane under distributed load

$$\nabla^2 u(\mathbf{x}) = -\frac{1}{T} g(\mathbf{x}), \quad \mathbf{x} \in \Omega \subset \mathbb{R}^2$$
$$u(\mathbf{x}) = 0, \quad \mathbf{x} \in \Gamma$$

Fundamental solution

$$F(\mathbf{x} - \mathbf{x}_0) = \frac{\ln 1/|\mathbf{x} - \mathbf{x}_0|}{2\pi}$$

Singularity at $r = 0$
Membrane cannot bear a concentrated force



Boundary Integrals

Test with testing function $\psi(x)$

$$\int_\Omega \psi(\mathbf{x})\nabla^2 u(\mathbf{x})\mathrm{d}\mathbf{x} = -\frac{1}{T}\int_\Omega \psi(\mathbf{x})g(\mathbf{x})\mathrm{d}\mathbf{x}$$

Integrate by parts as long as the operator (second derivatives) is shifted to the testing function

$$\int_\Omega \psi(\mathbf{x})\nabla^2 u(\mathbf{x})\mathrm{d}\mathbf{x} = \int_\Gamma \psi(\mathbf{x})\frac{\partial u(\mathbf{x})}{\partial n}\mathrm{d}\mathbf{x} - \cancel{\int_\Gamma \frac{\partial \psi(\mathbf{x})}{\partial n} u(\mathbf{x})\mathrm{d}\mathbf{x}} + \int_\Omega \nabla^2\psi(\mathbf{x})u(\mathbf{x})\mathrm{d}\mathbf{x}$$

At each integration, one boundary integral is extracted from the volume integral. Finally, we get the original operator ∇^2 acting on the testing function.

Boundary Integral Representation

Make the choice $\psi = F$. As a result, the volume integral with the operator acting on F transforms into

$$\int_\Omega \nabla^2 F(\mathbf{x},\mathbf{x}_0)u(\mathbf{x})\mathrm{d}\mathbf{x} = -\int_\Omega \delta(\mathbf{x}-\mathbf{x}_0)u(\mathbf{x})\mathrm{d}\mathbf{x} = \begin{cases} -u(\mathbf{x}_0) & \mathbf{x}_0 \in \Omega \\ -\frac{1}{2}u(\mathbf{x}_0) & \mathbf{x}_0 \in \Gamma \\ 0 & \text{otherw.} \end{cases}$$

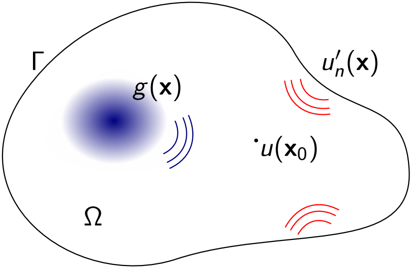
Substituting and rearranging leads to BIR:

$$\int_\Gamma F(\mathbf{x},\mathbf{x}_0)\frac{\partial u(\mathbf{x})}{\partial n}\mathrm{d}\mathbf{x} + \frac{1}{T}\int_\Omega F(\mathbf{x},\mathbf{x}_0)g(\mathbf{x})\mathrm{d}\mathbf{x} = \begin{cases} u(\mathbf{x}_0) & \mathbf{x}_0 \in \Omega \\ \frac{1}{2}u(\mathbf{x}_0) & \mathbf{x}_0 \in \Gamma \\ 0 & \text{otherw.} \end{cases}$$

Physical Interpretation

Physical interpretation by exploiting symmetry $F(\mathbf{x}, \mathbf{x}_0) = F(\mathbf{x}_0, \mathbf{x})$

$$\underbrace{\int_\Gamma F(\mathbf{x}_0, \mathbf{x}) \frac{\partial u(\mathbf{x})}{\partial n} \mathrm{d}\mathbf{x}}_{\text{surface convolution}} + \underbrace{\frac{1}{T} \int_\Omega F(\mathbf{x}_0, \mathbf{x}) g(\mathbf{x}) \mathrm{d}\mathbf{x}}_{\text{volume convolution}} = u(\mathbf{x}_0), \quad \mathbf{x}_0 \in \Omega$$



Volume convolution $\rightarrow u_{\mathrm{inc}}(\mathbf{x}_0)$
Source: $g(\mathbf{x})$

Surface convolution $\rightarrow u_{\mathrm{scat}}(\mathbf{x}_0)$
Source: $u'_n(\mathbf{x})$

$$u_{\mathrm{inc}}(\mathbf{x}_0) + u_{\mathrm{scat}}(\mathbf{x}_0) = u(\mathbf{x}_0)$$

Boundary Integral Equation

Let $\mathbf{x}_0$ approach the boundary Γ

$$u_{\mathrm{inc}}(\mathbf{x}_0) + u_{\mathrm{scat}}(\mathbf{x}_0) = 0$$

$$\int_\Gamma F(\mathbf{x}_0, \mathbf{x}) \frac{\partial u(\mathbf{x})}{\partial n} \mathrm{d}\mathbf{x} = -u_{\mathrm{inc}}(\mathbf{x}_0)$$

This BIE is solved numerically

Boundary Discretisation

Discretise the boundary into boundary elements $\Gamma = \bigcup \Gamma_e$

$$\sum_{e=1}^{E} \int_{\Gamma_e} F(\mathbf{x}_0, \mathbf{x}) \frac{\partial u(\mathbf{x})}{\partial n} \mathrm{d}\mathbf{x} = -u_{\mathrm{inc}}(\mathbf{x}_0)$$

Discretise the unknown Neumann data:
Take the normal derivative of the displacement $\partial u / \partial n = \phi_e$ as constant over Γ_e

$$\sum_{e=1}^{E} \int_{\Gamma_e} F(\mathbf{x}_0, \mathbf{x}) \mathrm{d}\mathbf{x}\, \phi_e = -u_{\mathrm{inc}}(\mathbf{x}_0)$$

Collocation: Write E independent equations by placing the source point $\mathbf{x}_0$ into the center of each element $\mathbf{x}_i$, $i = 1, \ldots E$

$$\sum_{e=1}^{E} \int_{\Gamma_e} F(\mathbf{x}_i, \mathbf{x}) \mathrm{d}\mathbf{x}\, \phi_e = -u_{\mathrm{inc}}(\mathbf{x}_i), \quad i = 1 \ldots E$$

BEM System of Equations

$$\mathbf{F}\phi = \mathbf{g}$$

$$F_{ij} = \int_{\Gamma_j} F(\mathbf{x}_i, \mathbf{x})\mathrm{d}\mathbf{x}$$

$$g_i = -u_{\text{inc}}(\mathbf{x}_i) = -\frac{1}{T}\int_{\Omega} F(\mathbf{x}_i, \mathbf{x})g(\mathbf{x})\mathrm{d}x$$

System Matrix **F**

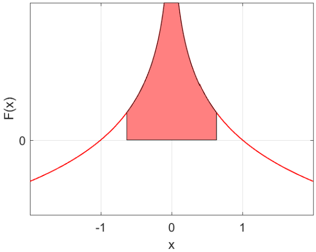
- Full
- Not Symmetric
- Elements computed by (numerically) integrating the Fundamental Solution

Singular Integrals

As the fundamental solution is singular, the diagonal matrix elements need to be handled separately

Simple cases analytical integration

$$
\begin{aligned}
F_{ii} &= \int_{\Gamma_i} F(\mathbf{x}_i, \mathbf{x}) \mathrm{d}\mathbf{x} \\
&= \int_{-d/2}^{d/2} \frac{\ln 1/|x|}{2\pi} \mathrm{d}x \\
&= \frac{d}{2\pi} (1 - \ln d/2)
\end{aligned}
$$



Off-diagonal elements can be computed numerically