Kirchhoff-type modelling of concentrated parameter mechanical systems

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Engineering Acoustics Lecture Notes
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Elements of mechanical systems

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Concentrated mass

Assumptions:

- acceleration proportional to excitation force

Newton’s second law

\[ f(t) = ma(t) \]  \hspace{2cm} (1)

where

- \( t \) is time [s]
- \( f \) is excitation force [N]
- \( a \) is acceleration [m/s^2]
- \( m \) is mass [kg]

Note: acceleration is measured relative to a *fixed* reference position (infinite mass)
Concentrated stiffness/compliance

Assumptions:
- deformation proportional to excitation force

Hooke's law

\[ f(t) = ku(t) = \frac{1}{c} u(t) \]  (2)

where

- \( f \) is excitation force [N]
- \( u \) is linear deformation \((L(t) - L_0)\) [m]
- \( k \) is stiffness [N/m]
- \( c \) is compliance [m/N]
Concentrated stiffness /compliance

Hooke’s law in more general form:

$$\sigma(t) = E \epsilon(t)$$  \hspace{1cm} (3)

where

- $\sigma$ is mechanical stress (force per unit area) $[\text{N/m}^2]$
- $\epsilon$ is strain $\left(\frac{L-L_0}{L_0}\right) [-]$
- $E$ is Young’s modulus of elasticity $[\text{N/m}^2]$ – material parameter

Relation of stiffness / compliance to material and geometrical parameters:

$$k = \frac{EA}{L_0}, \quad c = \frac{L_0}{EA}$$  \hspace{1cm} (4)
Concentrated viscous damping

Assumptions:

- velocity proportional to excitation force

\[ f(t) = rv(t) \]  

where

- \( f \) is excitation force [N]
- \( v \) is velocity of deformation \( \frac{d}{dt}(L(t) - L_0) \) [m/s]
- \( r \) is viscous damping [Ns/m]
Mechanical impedance

Impedance is interpreted in frequency domain: 
\[ f(t) = \hat{f}(\omega) e^{j\omega t} \]

Assumptions:

- velocity proportional to excitation force

\[ \hat{f}(\omega) = z_m(\omega) \hat{\nu}(\omega) \]  

\[ \hat{f} \] is complex amplitude of excitation force \([N]\)
\[ \hat{\nu} \] is complex amplitude of velocity of deformation \([m/s]\)
\[ z_m \] is mechanical impedance \([Ns/m]\)
Mechanical impedance of concentrated elements

- mechanical impedance of mass:
  \[ z_{\text{mass}} = \frac{\hat{f}}{\hat{v}} = \frac{\hat{f}}{\frac{\hat{a}}{j\omega}} = j\omega m \]  
  (7)

- mechanical impedance of compliance:
  \[ z_{\text{comp}} = \frac{\hat{f}}{\hat{v}} = \frac{\hat{f}}{\frac{1}{j\omega \hat{u}}} = \frac{1}{j\omega c} \]  
  (8)

- mechanical impedance of damping:
  \[ z_{\text{damp}} = \frac{\hat{f}}{\hat{v}} = r \]  
  (9)
Dynamic mechanical transfer quantities

\( \hat{f} / \hat{u} \): dynamic stiffness  \( \hat{u} / \hat{f} \): dynamic compliance
\( \hat{f} / \hat{v} \): impedance  \( \hat{v} / \hat{f} \): admittance
\( \hat{f} / \hat{a} \): dynamic mass  \( \hat{a} / \hat{f} \): dynamic mobility
Connecting mechanical impedances

Common velocity

\[ \hat{v} = \hat{v}_1 = \hat{v}_2 \]  
\[ \hat{f}_1 = \hat{f}_2 \]  
\[ z_1 = z_2 \]

Common force

\[ \hat{f} = \hat{f}_1 = \hat{f}_2, \quad (13) \]
\[ \hat{v} = \hat{v}_1 + \hat{v}_2, \quad (14) \]
\[ z = z_1 \times z_2 \quad (15) \]

\[ \hat{v} = \hat{v}_1 = \hat{v}_2, \quad (10) \]
\[ \hat{f} = \hat{f}_1 + \hat{f}_2, \quad (11) \]
\[ z = z_1 + z_2 \quad (12) \]
\[ \hat{v}(s) = \hat{f}(s) \cdot \frac{1}{s m + r + \frac{1}{s c}} = \hat{f}(s) \cdot \frac{sc}{1 + \frac{1}{Q} \left( \frac{s}{\omega_0} \right) + \left( \frac{s}{\omega_0} \right)^2} \]  

(16)

with natural frequency \( \omega_0 \) and quality factor \( Q \) written as

\[ \omega_0 = \frac{1}{\sqrt{m c}}, \quad Q = \frac{\sqrt{m}}{r \sqrt{c}} \]  

(17)

or equivalently

\[ \hat{u}(s) = \frac{\hat{v}(s)}{s} = \hat{f}(s) c \cdot \frac{1}{1 + \frac{1}{Q} \left( \frac{s}{\omega_0} \right) + \left( \frac{s}{\omega_0} \right)^2} \]  

(18)
SDOF damped oscillator

\[
\frac{\hat{u}(s)}{f(s) \cdot c} = \frac{1}{1 + \frac{1}{Q} \frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2} = \frac{1}{1 + \frac{1}{Q} \frac{j\omega}{\omega_0} - \left(\frac{\omega}{\omega_0}\right)^2}
\] (19)

- **low frequency**: \( \omega < \omega_0 \)
  Transfer \( \approx 1 \)

- **mid frequency**: \( \omega \approx \omega_0 \)
  Transfer determined by quality factor \( Q \) (damping)
  Amplification at \( \omega = \omega_0 \): \( Q \)

- **high frequency**: \( \omega > \omega_0 \)
  Transfer \( \approx 1/\omega^2 \)
  Asymptotically: \(-12\,\text{dB per octave} \) (-40 dB per decade)
SDOF damped oscillator

Response to Dirac delta force excitation \( \hat{f}(s) = 1 \) (with damping factor \( \xi = 1/2Q \)):

\[
\hat{u}(s) = c \cdot \frac{1}{1 + \frac{1}{Q} \frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2} = c \cdot \frac{1}{1 + 2\xi \frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2}
\] (20)

poles of transfer function

\[
s = -\xi\omega_0 \pm j\omega_0 \sqrt{1 - \xi^2}
\] (21)

Time domain solution (with inverse Laplace transform):

\[
u(t) = \frac{c\omega_0}{\sqrt{1 - \xi^2}} e^{-t/\tau} \sin(\omega_d t)
\] (22)

with time constant and damped natural frequency written as

\[
\tau = \frac{1}{\omega_0 \xi}, \quad \omega_d = \omega_0 \sqrt{1 - \xi^2}
\] (23)
Vehicle with suspension

\[ \textbf{\( v_w \)} \] velocity of wheel

\[ \textbf{\( v_c \)} \] velocity of car

\[ \textbf{\( m_w \)} \] mass of wheel

\[ \textbf{\( c_s, r_s \)} \] compliance and damping of suspension

\[ \textbf{\( m_c \)} \] mass of car
## Mechano-electrical analogies

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</table>
Vehicle with suspension

\[ \frac{v_c}{v_w} = \frac{r_s + \frac{1}{sc_s}}{r_s + \frac{1}{sc_s} + smc} = \frac{1 + \frac{1}{Q \omega_0}}{1 + \frac{1}{Q \omega_0} + \left(\frac{s}{\omega_0}\right)^2} \]  

(24)

\[ \omega_0 = \frac{1}{\sqrt{mcsc}}, \quad Q = \frac{\sqrt{mc}}{r_s \sqrt{cs}} \]  

(25)
A dynamic interaction problem

- velocity generator $v_g$ with finite impedance $z_g$ loaded by impedance $z_l$

State I – disconnected
- generator vibrates with velocity $v_g$
- structure stands still $v = 0$

State II – connected
- both components vibrate with velocity $v$
- velocities modified by common interaction (contact) force $f_c$

$$f_c = z_g \cdot (v_g - v) = z_l \cdot v \quad (26)$$

$$f_c = v_g (z_g \times z_l), \quad v = \frac{f_c}{z_l} = \frac{z_g}{z_g + z_l} \quad (27)$$
Vinyl pickup

mechanical circuit

mechano-electrical analog
Vinyl pickup

Transfer at low frequencies: \( 1/\omega c_0 \gg \omega (m_s + m_a) \)

\[
\frac{v_s}{v_0} = \frac{sm_a}{sm_a + r_s + \frac{1}{sc_s}} = \frac{\left(\frac{s}{\omega_1}\right)^2}{1 + \frac{1}{Q_1} \left(\frac{s}{\omega_1}\right) + \left(\frac{s}{\omega_1}\right)^2}
\]

where

\[
\omega_1 = \frac{1}{\sqrt{m_a c_s}}, \quad Q_1 = \frac{\sqrt{m_a}}{r_s \sqrt{c_s}}
\]

Transfer at high frequencies: \( 1/\omega c_s \ll r_s, \omega m_a \gg r_s \)

\[
\frac{v_s}{v_0} = \frac{1}{\frac{1}{sc_0} + sm_s + r_s} = \frac{1}{1 + \frac{1}{Q_2} \left(\frac{s}{\omega_2}\right) + \left(\frac{s}{\omega_2}\right)^2}
\]

where

\[
\omega_2 = \frac{1}{\sqrt{m_s c_0}}, \quad Q_2 = \frac{\sqrt{m_s}}{r_s \sqrt{c_0}}
\]
Vinyl pickup

Parameters:

- $m_s = 0.1 \text{ g}$
- $\omega_1 = 2\pi \cdot 10 \text{ Hz}$, $Q_1 = 0.5$
- $m_a = 50 \text{ g}$
- $\omega_2 = 2\pi \cdot 10 \text{ kHz}$, $Q_2 = 1$
Tuned mass dampers

- Mass-spring oscillator \((M, C)\) excited by wideband force \(\hat{f}(s)\)
- Velocity response \(\hat{v}(s)\) in frequency domain

\[
\hat{v}(s) = \hat{f}(s) \cdot \frac{sC}{1 + \left(\frac{s}{\omega_0}\right)^2}
\]  

where \(\omega_0 = \frac{1}{\sqrt{MC}}\)

- Tuned mass damper: extend the system by an other mass-spring oscillator \((m, c)\) tuned to the same frequency

\(\omega_0 = \frac{1}{\sqrt{mc}}\)
Tuned mass dampers

**Mechanical Circuit**

**Mechano-electrical Analog**

**Input Impedance without TMD** \((m = 0)\)

\[
z = \frac{1}{sC} + sM = \frac{1 + s^2 MC}{sC} \quad \text{(33)}
\]

**Input Impedance with TMD**

\[
z = \frac{1}{sC} + sM + \frac{1}{sc} \times sm = \frac{1 + s^2 MC}{sC} + \frac{sm}{1 + s^2 mc} \quad \text{(34)}
\]
Tuned mass dampers

- $M/m = 10$
- $Q = q = \infty$

- $M/m = 10$
- $Q = q = 10$