The Finite Element Method in Acoustics
Excercises

Simulation Methods in Acoustics
07.11.2017.

Finite element system matrices in 1D

1. Write a function that assembles the acoustic finite element system matrices **K** and **M** of a one-dimensional system, having N elements (and $N+1$ nodes). Use piecewise linear shape functions. (You can assume that the ith element is defined by the ith and $i+1$-th nodes.)

   Reminder:[1]

   $$\mathbf{K}_e = \frac{\rho_0 c^2}{h_e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \mathbf{M}_e = \frac{\rho_0 h_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{1}$$

   Your function can take the vector of nodal locations, the equilibrium density ρ_0 and the speed of sound c as input.

[1] You can verify this result using the symbolic math toolbox.

1D finite element system

1. Create a simple geometry having 100 elements of equal length along the interval $x \in [0, L]$ with $L = 3\,\mathrm{m}$.
2. Use the function `[rho0, c] = air_constants(T)` to get the material properties ρ_0 and c at room temperature $T = 20^\circ\mathrm{C}$.
3. Compute the system matrices **K** and **M** of this system using the function you created previously. Check the structure of the system matrices using the function `spy`.
4. Compute the total sum of the elements of **K** and **M**, respectively. (Use e.g. `sum(sum(M))`)
   Does the results match your expectations?
5. Repeat the previous step, but perturb the nodal locations, such that the elements no longer have the same lengths.
   What is the result? Why?

1D finite element solution

(Use the system assembled in the previous steps.)

1. Compute the solution vector $\mathbf{p}$ with assuming the following Neumann BCs: $\bar{v}(0) = 1$, $\bar{v}(L) = 0$. Set the angular frequency as $\omega = 400\pi \,\mathrm{rad/s}$.

   Reminder:
   $$(\mathbf{K} - \omega^2 \mathbf{M})\mathbf{p} = -\mathrm{j}\omega \mathbf{q}$$

   Hint: in 1D and linear shape functions we can directly compute $\mathbf{q} = \rho_0^2 c^2 \left[-\bar{v}(0), 0, \ldots, 0, \bar{v}(L)\right]^{\mathrm{T}}$.
2. Plot the solution! (Note that the solution vector is complex!) Does the solution satisfy the prescribed boundary conditions?
3. Solve the system also with Dirichlet BC at $x = 0$: $\bar{p}(0) = 1$. (Keep $\bar{v}(L) = 0$ as above.)
   Hint: rearrange the system such that $v(0)$ is an unknown, and partition the vector $\mathbf{p}$ to $\mathbf{p}_{\mathrm{known}}$ and $\mathbf{p}_{\mathrm{unknown}}$ subvectors.
   Verify that the result satisfies the boundary conditions.

1D finite element solution – Robin BC

(Use the system assembled in the previous steps.)

1. Define the Robin BC at $x = L$ as $p(L) = z_2 v(L)$. Verify (on paper) that the resulting system can be written in the form:

$$\left(\mathbf{K} + \mathrm{j}\omega\mathbf{C} - \omega^2\mathbf{M}\right)\mathbf{p} = -\mathrm{j}\omega\mathbf{q}$$

with $z_0 = \rho_0 c$ and

$$\mathbf{C} = z_0^2 \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & \frac{1}{z_2} \end{bmatrix}, \qquad \mathbf{q} = z_0^2 \begin{bmatrix} -\bar{v}(0) \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

2. Solve the system with $z_2 = z_0$ and examine the solution.
3. Solve the system with $z_2 = z_0/2$ and examine the solution.

1D finite element – modes

(Use the system matrices **K** and **M** assembled previously.)

1. Compute the first 15 modes of the finite element system

   ```
   [Phi, Lambda] = eigs(K, chol(M), nModes, 'sm',...
               struct('cholB', 1, 'symm', 1) );
   ```

2. Sort the modes by frequency in ascending order

   ```
   [Om, i] = sort(sqrt(diag(Lambda)));
   Phi = Phi(:,i);
   ```

3. Plot some of the mode shapes. Check the derivative of the mode shapes $\partial\phi/\partial x$ at $x = 0$ and $x = L$. Explain the result.
4. Compare the eigenfrequencies to each other. Does the result match your expectations? Hint: divide the eigenfrequencies by the first non-zero eigenfrequency.

1D finite element – inhomogeneous

1. Modify your system matrix assembly function such that the material properties ρ_0 and c can be set for each element individually. (Take the input parameters ρ_0 and c as vectors.)
2. Compute the material properties for each element assuming a linear dependency of the temperature with $T(0) = 20^{\circ}\mathrm{C}$ and $T(L) = 80^{\circ}\mathrm{C}$. (You can use the function `air_constants` on a vector input.)
3. Assemble the system matrices and compute the mode shapes, as in the previous step. Compare the mode shapes and the eigenfrequencies with that of the homogeneous case. What differences do you observe?

1D finite element – time domain

1. Solve the initial value problem in the time domain

$$\begin{aligned} v(x=0, t<1\mathrm{ms}) &= 1 \\ v(x=0, t\geq 1\mathrm{ms}) &= 0 \\ v(x=L, t) &= 0 \\ p(x, t=0) &= 0 \\ \dot{p}(x, t=0) &= 0 \end{aligned}$$

Apply the method of modal superposition.

$$\mathbf{p} = \boldsymbol{\Psi}\boldsymbol{\alpha}, \quad \boldsymbol{\Lambda}\boldsymbol{\alpha} + \ddot{\boldsymbol{\alpha}} = -\boldsymbol{\Psi}^{\mathrm{T}}\dot{\mathbf{q}}$$

The equivalent state space representation is

$$\dot{\mathbf{x}} = \begin{bmatrix} \mathbf{0} & -\boldsymbol{\Lambda} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \mathbf{x} + \begin{Bmatrix} -\boldsymbol{\Psi}^{\mathrm{T}}\dot{\mathbf{q}} \\ \mathbf{0} \end{Bmatrix}, \qquad \mathbf{x} = \begin{Bmatrix} \dot{\boldsymbol{\alpha}} \\ \boldsymbol{\alpha} \end{Bmatrix} \tag{2}$$

Solve the resulting system with a backward Euler iterative scheme