Plane wave theory

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Engineering Acoustics Lecture Notes

Wave equation in 1D

Euler's equation: Newton's second law for the sound particle

$$P(x,t)A - P(x+\Delta x,t)A = \rho_0 \Delta x A \ddot{u}(x,t) \tag{1}$$

Simplification and limit ($\Delta x \to 0$)

$$p'(x,t) + \rho \dot{v}(x,t) = 0 \tag{2}$$

Wave equation in 1D

Gas law for adiabatic state changes:

$$PV^{\kappa} = \text{const} = C \tag{3}$$

linearization around P_0 and V_0

$$\mathrm{d}P|_{V_0} = p = -\kappa P_0 \frac{\mathrm{d}V}{V_0} \tag{4}$$

Application for the sound particle:

$$p(x,t) = -\kappa P_0 \frac{u(x+\Delta x, t)A - u(x,t)A}{\Delta x A} \to -\kappa P_0 u'(x,t) \tag{5}$$

Wave equation in 1D

Combination of (2) and (5):

$$p''(x,t) + \frac{1}{c^2}\ddot{p}(x,t) = 0 \tag{6}$$

where

$$c = \sqrt{\frac{\kappa P_0}{\rho_0}} \tag{7}$$

is speed of sound.

Solution of the wave equation

d'Alembert's solution:

$$p(x,t) = p^{+}(x - ct) + p^{-}(x + ct) \tag{8}$$

where p^{+} is wave form propagating in the positive direction with velocity c.

Plane wave specific impedance

Assumption: plane wave propagation in a single direction: $p^- = 0$

$$p(x,t) = p^+(x - ct) \tag{9}$$

from Euler's equation

$$\dot{v}(x,t) = -\frac{1}{\rho_0} p^{+\prime}(x - ct) \tag{10}$$

temporal integration:

$$v(x,t) = \frac{1}{\rho_0 c} p^+(x - ct) = \frac{p(x,t)}{z_0} \tag{11}$$

where $z_0 = \rho_0 c$ is the specific impedace of the plane wave.

Sound intensity of the plane wave

Work done by the sound particle during motion

$$\Delta E = p(x,t) A v(x,t) \Delta t \tag{12}$$

Intensity: sound power over unit surface

$$I(x,t) = p(x,t) v(x,t) \tag{13}$$

Mean intensity:

$$\bar{I}(x,t) = \frac{1}{T} \int_{t-T}^{t} I(x,\tau) \mathrm{d}\tau \tag{14}$$

In plane wave:

$$\bar{I}(x,t) = \rho_0 v_{\mathrm{rms}}^2(x,t) = \frac{1}{\rho_0} p_{\mathrm{rms}}^2(x,t) \tag{15}$$

Reference levels

Referece level of sound pressure: $p_0 = 20\ \mu\mathrm{Pa}$
Reference level of velocity: $v_0 = \frac{p_0}{z_0} = 5 \cdot 10^{-8}\ \mathrm{m/s}$
Reference level of intensity: $I_0 = p_0 v_0 = 1 \cdot 10^{-12}\ \mathrm{W/m^2}$

Reflection of plane waves

Reflection from open termination ($p = 0$)

$$p(0, t) = p^{+}(0 - ct) + p^{-}(0 + ct) = 0 \tag{16}$$

$$p^{+}(-y) = -p^{-}(y) \tag{17}$$

reflection with opposite sign

Reflection of plane waves

Reflection from rigid termination ($v = 0$)
From Euler's equation: $v \equiv 0 \rightarrow \dot{v} \equiv 0 \rightarrow p' = 0$

$$p'(0,t) = p^{+,}(0 - ct) + p^{-,}(0 + ct) = 0 \tag{18}$$

$$p^{+,}(-y) = -p^{-,}(y) \tag{19}$$

reflection with same sign

Impedace of sound field in a wave guide

Input impedance of wave guide with general termination