Numerical Simulation. The Direct Stiffness Method

December 11, 2017

Numerical Simulation

- What is a numerical solution? – Most of the time *approximate* solutions
- Why do we need such thing?
- When analytical solutions are available? – Simple geometries, simple boundary conditions.
- Semi-analyitcal solutions: infinite sums, etc.
- Validation of the method is important
- Hybrid models are also possible – Combination of 1D and 3D methods, for example

Errors

Approximuate solution contains inherent errors:

1. Model errors – some phenomena are neglected: e.g. nonlinear effects in sound propagation (these effects are not included in the original, continuous equations)
2. Input errors – the input parameters are uncertain
3. Rounding or truncation error – the computer cannot represent numbers to infinite precisions
4. Discretization errors – e.g. approximation of derivatives by divided differences

Can a maximum error be determined?

Errors

We will often solve equations of the type $\mathbf{Ax} = \mathbf{b}$.

- Correct solution: $\mathbf{x}_0 = \mathbf{A}^{-1}\mathbf{b}$.
- Estimate solution: $\mathbf{x}_{\text{est}}$.
- Forward error: $\mathbf{x}_0 - \mathbf{x}_{\text{est}}$.

Forward error is often impossible to compute as we require the knowledge of the exact solution $\mathbf{x}_0$.

$\mathbf{x}_{\text{est}}$ is the solution of the modified system $\mathbf{Ax} = \mathbf{b}_{\text{est}}$.
It is useful to compute $\mathbf{b} - \mathbf{b}_{\text{est}}$, which is the *backward error* of the solution.

Errors

Can the forward error be guessed from the backward error?
The problem is *well-conditioned* if small backward error implies small forward error.
The problem is *ill-conditioned*, *sensitive*, or *stiff* if this is not true.
The *condition number* measures the ratio of the relative error of the forward and backward errors.

The Direct Stiffness Method – Objectives

Algorithmic assembly of system of equations describing the static deflection of structures consisting of linear spring elements.
Observe many properties of the Finite Element Method discussed later during the course.
Introduced here:

- Concentrated parameters
- Truss system reduced to springs
- Assembly alogrithm
- Stiffness matrix

Linear Spring Element

Take a bar with length L, cross section area A and elasticity modulus (Young modulus) E. The bar is pulled by a force f, and deforms.

What are our assumptions? Under what circumstances they hold?

1. The bar has no mass.
2. The bar does not bend. (Bending stiffness is infinite.)
3. Forces at the ends of the bar act in a single point.
4. Strains are small: $\Delta L/L \ll 1$
5. The system is linear. (elongation proportional to tensile force)

Spring Element – Formulation

Definition of stress σ

$$\sigma A = f \tag{1}$$

Hooke's law:

$$\epsilon E = \frac{\Delta L}{L} E = \sigma \tag{2}$$

$$f = \frac{EA}{L} d = Kd \tag{3}$$

f: tensile force, d: elongation.
1 Degree of freedom: d
Here K is the sole parameter

Nodes and Degrees of Freedoms

- Introducing two end nodes and their displacements u_2, u_1, relative to a local frame of reference.
- Introducing the joint forces f_1 and f_2 at the two end nodes

$$d = u_2 - u_1, \qquad f = f_2 = -f_1 \tag{4}$$

The element stiffness matrix relating the forces to the displacements

$$\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = K \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{5}$$

- Only 1 independent equation. Infinite number of solutions for $f_2 = -f_1$, no solution for $f_2 \neq -f_1$

Global Frame of Reference

- Transforming the equations into a global frame of reference

$$u_i = u_{xi}\cos\phi + u_{yi}\sin\phi, \qquad i = 1,2 \tag{6}$$

$$f_{xi} = f_i \cos\phi \tag{7}$$

$$f_{yi} = f_i \sin\phi, \qquad i = 1,2 \tag{8}$$

Matrix form:

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} c & s & & \\ & & c & s \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \end{Bmatrix}, \qquad \begin{Bmatrix} f_{x1} \\ f_{y1} \\ f_{x2} \\ f_{y2} \end{Bmatrix} = \begin{bmatrix} c & \\ s & \\ & c \\ & s \end{bmatrix} \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \tag{9}$$

Or in matrix notation

$$\mathbf{u}_\xi = \mathbf{T}\mathbf{u}_x, \quad \mathbf{f}_x = \mathbf{T}^{\mathrm{T}}\mathbf{f}_\xi \tag{10}$$

Connections

$$\sum_{j=1}^{M} \mathbf{f}_{ij} = \mathbf{f}_i \tag{11}$$

$\mathbf{f}_{ij}$: joint force of j-th spring in i-th node. $\mathbf{f}_i$: external force excitation at node i

Structure of stiffness matrix

Symmetric: $\mathbf{K}^{\mathrm{T}} = \mathbf{K}$ – Reciprocity.

$$K_{ij} = \frac{f_i}{u_j} \qquad \text{if} \quad u_l = 0 \quad (l \neq j) \tag{12}$$

$\mathbf{K}$ is positive semidefinite.

This can be proven by calculating the potential energy stored by the spring elements. For one spring we have:

$$E_{\mathrm{pot}} = \int_0^u f(y)\,\mathrm{d}y = \int_0^u Ky\,\mathrm{d}y = \frac{1}{2}Ku^2. \tag{13}$$

Summing for all the springs we can write

$$\begin{aligned} E_{\mathrm{pot}} &= \sum_i \int_0^{u_i} f_i(y_i)\,\mathrm{d}y_i = \int_0^{\mathbf{u}} \mathrm{d}\mathbf{y}^{\mathrm{T}}\mathbf{K}\mathbf{y} = \frac{1}{2}\int_0^{\mathbf{u}} \left(\mathrm{d}\mathbf{y}^{\mathrm{T}}\mathbf{K}\mathbf{y} + \mathbf{y}^{\mathrm{T}}\mathbf{K}\mathrm{d}\mathbf{y}\right) \\ &= \frac{1}{2}\int_0^{\mathbf{u}} \mathrm{d}\left(\mathbf{y}^{\mathrm{T}}\mathbf{K}\mathbf{y}\right) = \frac{1}{2}\mathbf{u}^{\mathrm{T}}\mathbf{K}\mathbf{u} \end{aligned} \tag{14}$$

Summary – Direct Stiffness Method

Algorithmic assembly of the system of equations describing the statics of truss structures
System with N end points $\rightarrow 2N$ displacement DOF, u_{ix}, u_{iy}

$$\mathbf{K}\mathbf{u} = \mathbf{f} \tag{15}$$

$$\begin{bmatrix} K_{1x1x} & K_{1x1y} & K_{1x2x} & \dots & K_{1xNy} \\ K_{1y1x} & K_{1y1y} & K_{1y2x} & \dots & K_{1yNy} \\ K_{2x1x} & K_{2x1y} & K_{2x2x} & \dots & K_{2xNy} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ K_{Ny1x} & K_{Ny1y} & K_{Ny2x} & \dots & K_{NyNy} \end{bmatrix} \begin{Bmatrix} u_{1x} \\ u_{1y} \\ u_{2x} \\ \vdots \\ u_{Ny} \end{Bmatrix} = \begin{Bmatrix} f_{1x} \\ f_{1y} \\ f_{2x} \\ \vdots \\ f_{Ny} \end{Bmatrix} \tag{16}$$

f: vector of nodal external forces (excitation)
u: vector of nodal displacements (response)

Summary – Steps of matrix assembly

- Element-by-element (spring-by-spring) assembly
- Spring parameters: Stiffness: $K = EA/L$, inclination: ϕ
- Hooke's law in local (ξ, η) coordinate system:

$$\underbrace{K \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\mathbf{K}_{e\xi}} \begin{Bmatrix} u_{1\xi} \\ u_{1\eta} \\ u_{2\xi} \\ u_{2\eta} \end{Bmatrix} = \begin{Bmatrix} f_{1\xi} = -f_{2\xi} \\ f_{1\eta} = 0 \\ f_{2\xi} \\ f_{2\eta} = 0 \end{Bmatrix} \tag{17}$$

- Hooke's law in global (x, y) coordinate system

$$\underbrace{\mathbf{T}^{\mathrm{T}} \mathbf{K}_{e\xi} \mathbf{T}}_{\mathbf{K}_e} \begin{Bmatrix} u_{ix} \\ u_{iy} \\ u_{jx} \\ u_{jy} \end{Bmatrix} = \begin{Bmatrix} f_{ix} \\ f_{iy} \\ f_{jx} \\ f_{jy} \end{Bmatrix}, \quad \mathbf{T} = \begin{bmatrix} c & s & & \\ -s & c & & \\ & & c & s \\ & & -s & c \end{bmatrix} \tag{18}$$

Summary – Steps of matrix assembly

- Total matrix assembly

$$\mathbf{K} = \sum \mathbf{K}_e \tag{19}$$

$$\mathbf{K}_e = \begin{bmatrix} & & & & & & \\ & x & x & & x & x & \\ & x & x & & x & x & \\ & & & & & & \\ & x & x & & x & x & \\ & x & x & & x & x & \\ & & & & & & \end{bmatrix} \tag{20}$$