Finite Dimensional Function Approximations

Simulation Methods in Acoustics

Definitions

Objective: represent a continuous function in a finite dimensional space (with a finite number of coordinates)
Approximation of function $f(x)$

$$f(x) \approx \hat{f}(x) = \sum_{j=1}^{N} q_j \phi_j(x)$$

- $\phi_j(x)$: A priori known base functions
- q_j: coefficents (coordinates)
- Why linear formula?
  Coordinates of $f(x) + g(x)$ should be $q_j + w_j$
- Objective: find q_j, $j = 1 \dots N$ so that the approximation error (residual) $r(x)$ is small.

$$r(x) = f(x) - \hat{f}(x)$$

Collocation

Collocation: N constraints defined by values of the residual function: $r(x_i) = 0,\ i = 1 \ldots N$

$$r(x_i) = 0 \rightarrow \sum_{j=1}^{N} q_j \phi_j(x_i) = f(x_i)$$

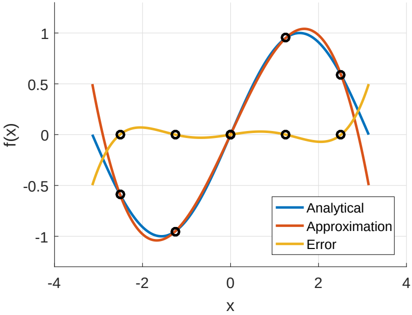
Results in system of linear equations

$$\left[\phi_j(x_i)\right] \left\{q_j\right\} = \left\{f(x_i)\right\}$$

Solution by matrix inversion

Collocation Example

$$f(x) = \sin(x)$$
$$\phi_j(x) = \{1, x, x^2, x^3, \dots\}$$
$$x_i =: \text{equidistant over a single period}$$
$$\phi_j(x_i) : \text{full Vandermonde matrix}$$



$$\hat{f}(x) = 0.9311x - 0.1104x^3$$

Collocation – How to select collocation points?

- Solution is easy if system matrix is close to diagonal

$$\phi_j(x_i) = 0, \text{ if } i \neq j$$

- Typical local examples:
  - Piecewise constant:

$$\phi_i(x) = \begin{cases} 1 & x \in \Omega_i \\ 0 & \text{otherwise} \end{cases} \qquad x_i \in \Omega_i$$

  - Piecewise linear (hat functions)
- Typical global examples:
  - Sinc interpolation:

$$\phi_i(x) = \mathrm{sinc}\left(\frac{x - i \cdot \Delta x}{\pi}\right), \qquad x_i = i \cdot \Delta x$$

Generalized Collocation

- Generalized condition on the samples of the residual's derivatives
$$r^{(n_i)}(x_i) = 0$$

Galerkin method

Objective: L2 norm of the residual should be minimal

$$\|r\|^2 = \langle r, r \rangle = \min$$

$$\frac{\partial}{\partial q_i} \langle r, r \rangle = 0, \quad i = 1 \ldots N$$

$$2 \left\langle \frac{\partial}{\partial q_i} r, r \right\rangle = 0, \quad i = 1 \ldots N$$

$$\frac{\partial}{\partial q_i} r = -\frac{\partial}{\partial q_i} \hat{f} = -\frac{\partial}{\partial q_i} \sum_{j=1}^{N} q_j \phi_j(x) = -\phi_i$$

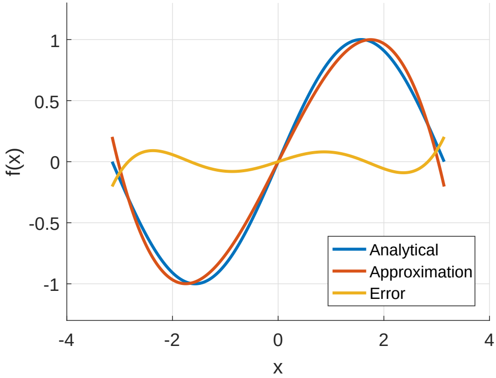
$$\langle \phi_i, r \rangle = 0, \quad i = 1 \ldots N$$

$$\sum_j \langle \phi_i, \phi_j \rangle q_j = \langle \phi_i, f \rangle, \quad i = 1 \ldots N$$

$$\left[\langle \phi_i, \phi_j \rangle \right] \left\{ q_j \right\} = \left\{ \langle \phi_i, f \rangle \right\}$$

Galerkin Example

$$f(x) = \sin(x)$$
$$\phi_j(x) = \{1, x, x^2, x^3, \dots\}$$
$$\langle \phi_i, \phi_j \rangle : \text{full matrix}$$



$$\hat{f}(x) = 0.8570x - 0.0934x^3$$

Galerkin

How to choose the base functions ϕ_i?

- Solution is easy if the base functions are orthogonal

$$\langle \phi_i, \phi_j \rangle = 0, \text{ if } i \neq j$$

- Typical local examples:
  - Piecewise constant:
- Typical global examples:
  - Orthogonal (Legendre) polynomials
  - Trigonometric polynomials

Galerkin – Piecewise Constant Base Functions

Equidistant rect functions, $x_i = h \cdot i$

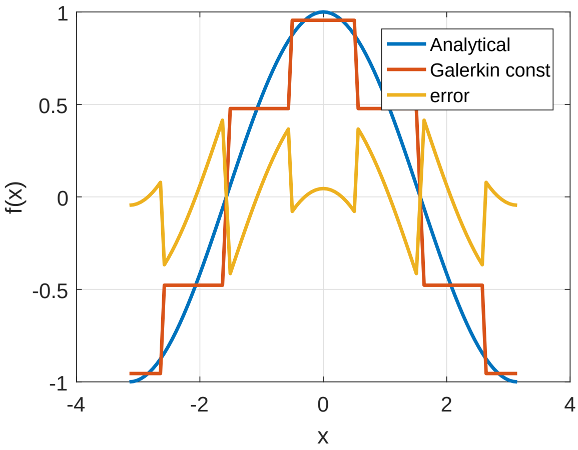
$$\phi_i(x) = H(\frac{x - x_i}{h})$$

$$< \phi_i, \phi_j >= \delta_{ij} h$$

$$\mathbf{A} = h \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{bmatrix} \quad \mathbf{b} = \left\{ \int_{x_i - h/2}^{x_i + h/2} f(x) \mathrm{d}x \right\}$$

$$q_i = \frac{1}{h} \int_{x_i - h/2}^{x_i + h/2} f(x) \mathrm{d}x$$

Galerkin – Piecewise Constant Example



Galerkin – Piecewise Linear Base Functions

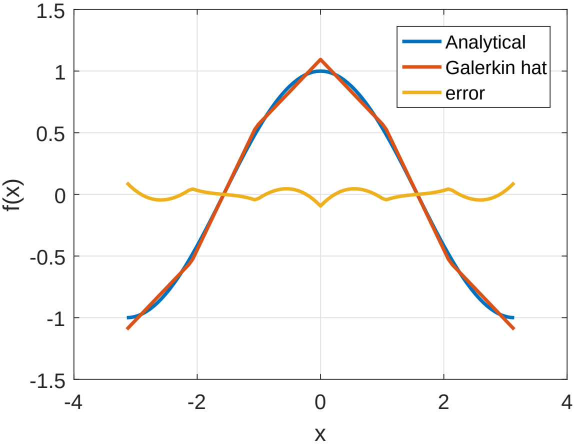
Equidistant hat functions, $x_i = h \cdot i$

$$\phi_i(x) = \begin{cases} 1 + \frac{x - x_i}{h} & -h + x_i \leq x \leq x_i \\ 1 - \frac{x - x_i}{h} & x_i \leq x \leq h + x_i \\ 0 & \text{otherwise} \end{cases}$$

$$< \phi_i, \phi_i >= \frac{2h}{3}, \qquad < \phi_i, \phi_{i+1} >= \frac{h}{6}$$

$$\mathbf{A} = \frac{h}{6} \begin{bmatrix} 2 & 1 & & & \\ 1 & 4 & 1 & & \\ & 1 & 4 & 1 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

Galerkin – Piecewise Linear Example



Galerkin – Trigonometric Polynomials

$$\phi_i(x) = \sin\left(\frac{i\pi}{L}x\right), \qquad 0 \le x \le L$$

$$\langle \phi_i, \phi_j \rangle = \delta_{ij}\frac{L}{2}$$

$$q_i = \frac{\langle \phi_i, f \rangle}{\|\phi_i\|^2} = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{i\pi}{L}x\right)\mathrm{d}x$$

Conclusion: Galerkin method with trigonometric polynomial base results in Fourier series approximation

Weighted Residual Method

The Collocation and Galerkin methods can be generalized as the method of weighted residuals.

$$f(x) \approx \hat{f}(x) = \sum_{j=1}^{N} q_j \phi_j(x)$$

Where the residual is weighted by the test function ψ_i

$$\langle \psi_i, r \rangle = 0, \quad i = 1 \ldots N$$

- ϕ_i: trial function, shape function
- ψ_i: test function

- Collocation: $\psi_i(x) = \delta(x - x_i)$
- Galerkin: $\psi_i(x) = \phi_i(x)$