Weak form of Boundary Value Problems

Simulation Methods in Acoustics

Note on finite dimensional description of functions

- Approximation:

$$f(x) \approx \hat{f}(x) = \sum_{j=1}^{N} q_j \phi_j(x)$$

- Residual function: $r(x) = f(x) - \hat{f}(x)$
- Collocation and Galerkin with test functions $\psi_i(x)$

$$\langle \psi_i, r \rangle = 0 \quad \text{for all } i = 1, 2 \ldots N$$

  - Collocation: $\psi_i(x) = \delta(x_i)$
  - Galerkin: $\psi_i(x) = \phi_i(x)$
- Other ψ – ϕ choices are possible, but not used often in practice

The boundary value problem

- Given a BVP (PDE + BCs)

$$\mathcal{A}\{u(x)\} = f(x) \quad x \in \Omega \tag{1}$$

$$\mathcal{B}_i\{u(x)\} = \bar{g}_i(x) \quad x \in \Gamma_i \quad \bigcup_i \Gamma_i = \Gamma \tag{2}$$

- $\mathcal{A}$ and $\mathcal{B}_i$ are differential operators
- Example (Poisson equation)

$$-\frac{\partial^2 u(x)}{\partial x^2} = 1 \qquad x \in \Omega = [0, 1] \tag{3}$$

$$u(x) = 0 \qquad x = 0 \quad \text{(Dirichlet)} \tag{4}$$

$$\frac{\partial u(x)}{\partial x} = -1 \qquad x = 1 \quad \text{(Neumann)} \tag{5}$$

- In our example:
  - $\mathcal{A} = -\partial^2/\partial x^2$, $f(x) \equiv 1$
  - $\mathcal{B}_1 = 1$, $\bar{g}_1(x) \equiv 0$
  - $\mathcal{B}_2 = \partial/\partial x$, $\bar{g}_2 \equiv -1$

Weak form construction

- Construction steps:
  1. Multiply PDE by a "sufficiently well behaved"[1] function $\psi(x)$
  2. Integrate over the whole domain Ω
  3. Integration by parts as much as possible (shift derivatives)
  4. Apply BC

1. Multiply the PDE by the test function $\psi(x)$

$$\psi(x)\mathcal{A}\{u(x)\} = \psi(x)f(x) \qquad \forall x \in \Omega$$

2. If the equality holds, the integrals must be also equal

$$\int_\Omega \psi(x)\mathcal{A}\{u(x)\}\,\mathrm{d}x = \int_\Omega \psi(x)f(x)\,\mathrm{d}x \qquad \forall\psi(x)$$

- Statement: The solution of the weak form must also be the solution of the strong (original) form of the PDE
- Proof: The weak form must hold for all $\psi(x)$. We can choose test functions that are nonzero only in $B_\delta(x_0)$ for any $x_0 \in \Omega$. Thus, the original equation must hold for all $x_0 \in \Omega$.
- Then, why it is a *weak* form?

[1] The definition well behaved depends on the type of PDE and BC

Weak form example I.

- Look at the Poisson equation example:

$$-\frac{\partial^2 u(x)}{\partial x^2} = f(x) \quad x \in \Omega = [0,1]\,, f(x) \equiv 1$$

1. Multiply by the test function $\psi(x)$

$$-\psi(x)\frac{\partial^2 u(x)}{\partial x^2} = \psi(x)f(x)$$

2. The weak form is written as

$$\int_\Omega -\psi(x)\frac{\partial^2 u(x)}{\partial x^2}\,\mathrm{d}x = \int_\Omega \psi(x)f(x)\,\mathrm{d}x$$

3. Apply integration by parts on the l.h.s.
   - Recall that $\int_{x_1}^{x_2} f'g\,\mathrm{d}x = [fg]_{x_1}^{x_2} - \int_{x_1}^{x_2} fg'\,\mathrm{d}x$
   - With $f = \partial u/\partial x$ and $g = \psi(x)$

$$-\left[\psi(x)\frac{\partial u(x)}{\partial x}\right]_0^1 + \int_0^1 \frac{\partial \psi(x)}{\partial x}\frac{\partial u(x)}{\partial x}\,\mathrm{d}x = \int_0^1 \psi(x)f(x)\,\mathrm{d}x$$

Weak form example II.

4. Apply boundary conditions
   - $u(0) = 0$
   - $\frac{\partial u(x)}{\partial x}\Big|_{x=1} = -1$

$$\psi|_{x=1} + \left(\psi \frac{\partial u}{\partial x}\right)\Big|_{x=0} + \int_0^1 \frac{\partial \psi}{\partial x} \frac{\partial u}{\partial x} \,\mathrm{d}x = \int_0^1 \psi(x) 1 \,\mathrm{d}x$$

- Let's compare the strong and weak forms
  - Strong: 2nd derivative of u, Weak: 1st
  - Strong: f continuous on Ω, Weak: integrable ($f\psi$ integrable)
- The weak form imposes less strict criteria on u and f! But it requires some conditions for $\psi(x)$.
- We can express these criteria using function space definitions:
  - Strong: $u \in C^2(\Omega)$, $f \in C^0(\Omega)$
  - Weak: $u \in C^1(\Omega)$, $f \in C^{-1}(\Omega)$, $u, \partial u/\partial x \in L^2(\Omega)$

Sobolev space definition

- Square-integrable functions (in Lebesgue sense)

$$L^2(\Omega) = \left\{ f : \Omega \to \mathbb{R} \,\middle|\, \|f\|^2 = \int_\Omega f^2 \,\mathrm{d}x < \infty \right\}$$

- Sobolev space $\sim$ the function and its derivatives are in L^p

$$H^1(\Omega) = \left\{ f \in L^2(\Omega), \frac{\partial f}{\partial x} \in L^2(\Omega) \right\}$$

- Note: derivatives are meant in a weak sense
  - $g(x)$ is the weak derivative of u, i.e. $g = \partial u / \partial x$ if and only if

$$\int_\Omega \psi(x) g(x) \,\mathrm{d}x = - \int_\Omega \frac{\partial \psi(x)}{\partial x} u(x) \,\mathrm{d}x \qquad \forall \psi(x) \in C_0^\infty(\Omega)$$

  - The weak derivative is unique
  - Note: $C_0^\infty(\Omega)$ means infinite times differentiable functions having a compact support on Ω

Example: solve the weak form

- Let's search the solution in the form:
  - $u(x) = Ax + Bx^2$
  - $\psi(x) = \hat{A}x + \hat{B}x^2$
- Find A, B such that the weak form is satisfied for all $\hat{A}$, $\hat{B}$
- Substitute:

$$\hat{A}x + \hat{B}x^2\Big|_{x=1} + \cancel{\left(\hat{A}x + \hat{B}x^2\right)\left(A + 2Bx\right)\Big|_{x=0}} + \int_0^1 \left(\hat{A} + 2\hat{B}x\right)\left(A + 2Bx\right)\,\mathrm{d}x = \int_0^1 \left(\hat{A}x + \hat{B}x^2\right)\,\mathrm{d}x$$

- Rearrange (A and B on l.h.s.)

$$\begin{bmatrix}\hat{A} & \hat{B}\end{bmatrix} \int_0^1 \begin{bmatrix}1 \\ 2x\end{bmatrix} \begin{bmatrix}1 & 2x\end{bmatrix}\,\mathrm{d}x \begin{bmatrix}A \\ B\end{bmatrix} = \begin{bmatrix}\hat{A} & \hat{B}\end{bmatrix} \left(\int_0^1 \begin{bmatrix}x \\ x^2\end{bmatrix}\,\mathrm{d}x - \begin{bmatrix}1 \\ 1\end{bmatrix} \right)$$

- Evaluating the integrals

$$\cancel{\begin{bmatrix}\hat{A} & \hat{B}\end{bmatrix}} \begin{bmatrix}1 & 1 \\ 1 & 4/3\end{bmatrix} \begin{bmatrix}A \\ B\end{bmatrix} = \cancel{\begin{bmatrix}\hat{A} & \hat{B}\end{bmatrix}} \begin{bmatrix}-1/2 \\ -2/3\end{bmatrix} \qquad \forall \begin{bmatrix}\hat{A} & \hat{B}\end{bmatrix}$$

- We obtain the solution: $A = 0, B = -1/2$, thus $u(x) = -x^2/2$

How did we solve?

- By choosing the trial functions ($u(x) = Ax + Bx^2$) the *trial function space was discretized* (x and x^2 are its two elements). Thus, we arrived at a finite dimensional problem (number of dimensions is $d = 2$ in this example)
- By letting $\psi(x) = \hat{A}x + \hat{B}x^2$ we used the Galerkin method.
- Note that we cannot apply the collocation method ($\psi_i = \delta(x_i)$) as $\partial\psi/\partial x$ must be in L^2
- By means of the discretization the BVP was reduced to an algebraic set of linear equations! Thus, in the end a system of linear equations was solved.
- In this case, we "luckily" got the analytical solution, however, in general we only get an *approximate solution*

Another solution

- Let's solve with a different right hand side:

$$-\frac{\partial^2 u(x)}{\partial x^2} = -\frac{1}{1+x^2} \qquad u(0)=0, \quad u'(1)=0$$

- All the same until we get

$$\begin{bmatrix}\hat{A} & \hat{B}\end{bmatrix} \begin{bmatrix}1 & 1\\ 1 & 4/3\end{bmatrix} \begin{bmatrix}A\\ B\end{bmatrix} = \begin{bmatrix}\hat{A} & \hat{B}\end{bmatrix} \int_0^1 \begin{bmatrix}\frac{-x}{1+x^2}\\ \frac{-x^2}{1+x^2}\end{bmatrix} \mathrm{d}x \qquad \forall \begin{bmatrix}\hat{A} & \hat{B}\end{bmatrix}$$

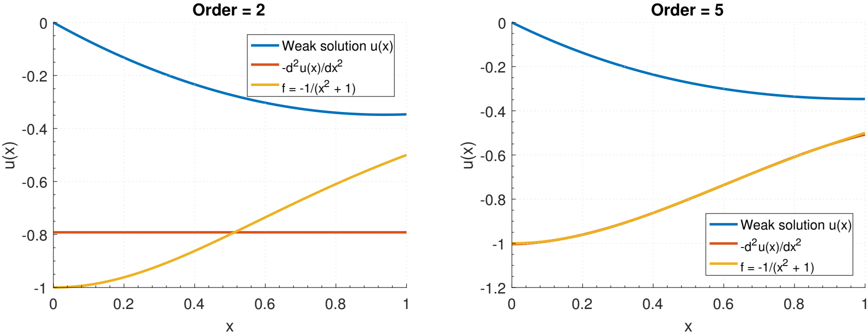
- By "dropping" $\begin{bmatrix}\hat{A} & \hat{B}\end{bmatrix}$

$$\begin{bmatrix}1 & 1\\ 1 & 4/3\end{bmatrix} \begin{bmatrix}A\\ B\end{bmatrix} = \begin{bmatrix}-\log(2)/2\\ \pi/4-1\end{bmatrix} \quad \rightarrow \quad \begin{bmatrix}A\\ B\end{bmatrix} = \begin{bmatrix}-0.7425\\ 0.3959\end{bmatrix}$$

- The discretized weak form gives an *approximate solution*

Visualization of the example

- The approximation gets better if we increase the dimensionality of our function space, e.g. by adding higher order polynomials



Beam example I.

- Solve the following BVP (static deflection of a clamped–free Euler–Bernoulli beam)

$$EIu''''(x) = f(x)$$
$$u(0) = 0 \qquad u'(0) = 0 \quad (Dirichlet)$$
$$u''(L) = 0 \qquad u'''(L) = 0 \quad (Neumann)$$

- Construct the weak form

$$EI \int_0^L \psi(x)u''''(x)\,\mathrm{d}x = \int_0^L \psi(x)f(x)\,\mathrm{d}x$$

  - Integration by parts once ...

$$EI \left([\psi u''']_0^L - \int_0^L \psi' u'''\,\mathrm{d}x \right) = \int_0^L f\psi\,\mathrm{d}x$$

  - ... and twice

$$EI \left([\psi u''']_0^L - [\psi' u'']_0^L + \int_0^L \psi'' u''\,\mathrm{d}x \right) = \int_0^L f\psi\,\mathrm{d}x$$

Beam example II.

- Choose the test and trial function spaces as

$$\phi_i = \psi_i = \sin\left(\frac{i\pi}{2L}\right) \qquad i = 1 \dots n$$

$$\phi_i = \psi_i = \cos\left(\frac{(i-n)\pi}{2L}\right) \qquad i = n+1 \dots 2n = N$$

- Note: by letting $\psi_i = \phi_i$ we use the Galerkin method
- The approximations using the above spaces reads as

$$u(x) \approx \hat{u}(x) = \sum_{j=1}^{N} q_j \phi_j(x) = \boldsymbol{\phi}(x)\mathbf{q}$$

$$\psi(x) \approx \hat{\psi}(x) = \sum_{j=1}^{N} w_j \phi_j(x) = \boldsymbol{w}^{\mathrm{T}} \boldsymbol{\phi}^{\mathrm{T}}(x)$$

- Note: $\boldsymbol{\phi}$ is a *row* vector, $\mathbf{q}$ and $\mathbf{w}$ are column vectors

Beam example III.

- Notice that the expressions

$$\left[\psi u'''\right]_0^L \qquad \left[\psi' u''\right]_0^L \qquad \int_0^L \psi'' u'' \,\mathrm{d}x$$

- Can all be written in a similar form

$$\mathbf{w}^{\mathrm{T}} \begin{bmatrix} \phi_1^{(\alpha)}\phi_1^{(\beta)} & \cdots & \phi_1^{(\alpha)}\phi_N^{(\beta)} \\ \vdots & \ddots & \vdots \\ \phi_N^{(\alpha)}\phi_1^{(\beta)} & \cdots & \phi_N^{(\alpha)}\phi_N^{(\beta)} \end{bmatrix} \begin{Bmatrix} q_1 \\ \vdots \\ q_N \end{Bmatrix}$$

- Integration $\int_0^L \mathrm{d}x$ or evaluation of $\phi_i^{(\alpha)}\phi_j^{(\beta)}$ (product of shape functions and their derivatives) at a fixed point is needed. Each term gives an $N \times N$ matrix in the square brackets.
- On the r.h.s. we have a column vector in the brackets

$$\mathbf{w}^{\mathrm{T}} \int_0^L \begin{bmatrix} \phi_1(x)f(x) \\ \vdots \\ \phi_N(x)f(x) \end{bmatrix} \mathrm{d}x = \mathbf{w}^{\mathrm{T}}\mathbf{b}$$

Beam example IV.

- Write the system in matrix notation:

$$EI\left[\mathbf{w}^{\mathrm{T}}\left(\mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3\right)\right]\mathbf{q} = \mathbf{w}^{\mathrm{T}}\mathbf{b}$$

  Note that this is a *scalar* equation
- This must hold for all vectors $\mathbf{w}^{\mathrm{T}}$, which is only possible if

$$EI\,\mathbf{A}\mathbf{q} = \mathbf{b}$$

- Thus, we arrive at a **system of linear algebraic equations**

Beam example V.

- Satisfying the BCs
  - Contrary to the previous example, our trial (shape) functions do not satisfy the Dirichlet BCs. ($\phi_i(0) \neq 0$, $\phi_i'(0) \neq 0$)
  - Constraints must be imposed:

$$\begin{bmatrix} \phi_1(0) & \phi_2(0) & \cdots & \phi_N(0) \\ \phi_1'(0) & \phi_2'(0) & \cdots & \phi_N'(0) \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ \vdots \\ q_N \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

  - The same in matrix form:

$$\mathbf{A}_\mathrm{c}\mathbf{q} = \mathbf{0}$$

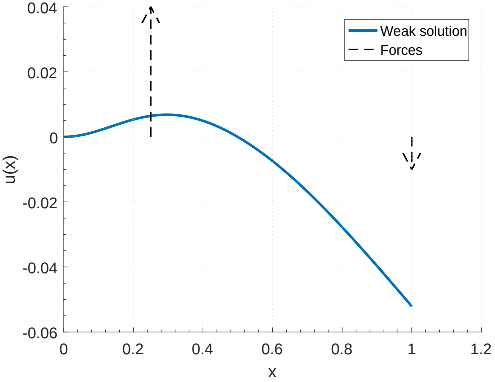
- We already know some methods for constraints ...
- ... let's use the Lagrange method

$$\begin{bmatrix} EI\mathbf{A} & \mathbf{A}_\mathrm{c}^\mathrm{T} \\ \mathbf{A}_\mathrm{c} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{q} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \mathbf{b} \\ \mathbf{0} \end{Bmatrix}$$

- This can finally be solved to get $\mathbf{q}$ and $\boldsymbol{\lambda}$

Beam – Solution

- The weak solution with $n = 5$ ($N = 10$) and two discrete loads as shown by the arrows in the figure



- Just to remember: the weak solution is a function

$$\hat{u}(x) = 0.6926 \sin\left(\frac{\pi x}{2}\right) + 0.1739 \sin(\pi x) - 0.7489 \sin\left(\frac{3\pi x}{2}\right) + 0.3287 \sin(2\pi x) - 0.0217 \sin\left(\frac{5\pi x}{2}\right)$$
$$- 0.8171 \cos\left(\frac{\pi x}{2}\right) + 1.388 \cos(\pi x) - 0.5373 \cos\left(\frac{3\pi x}{2}\right) - 0.0839 \cos(2\pi x) + 0.0503 \cos\left(\frac{5\pi x}{2}\right)$$