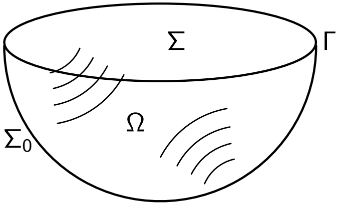
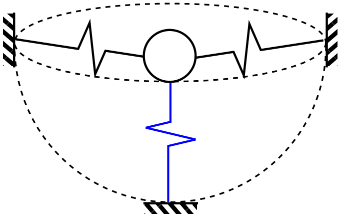
Vibroacoustic FEM/FEM Coupling

Simulation Methods in Acoustics

Introduction

Acoustic cavity inside a drum acts as a membrane stiffener.
Objective: Compute the stiffening effect by coupling a 2D mechanical FE model of the membrane to a 3D acoustic FE model of the enclosed cavity.





Coupled Problem

- Ideal membrane

$$-T\nabla^2 u(x,y) - \omega^2 \sigma u(x,y) = g(x,y), \qquad (x,y) \in \Sigma$$

$$u(x,y) = 0, \qquad (x,y) \in \Gamma$$

- Acoustic cavity

$$\begin{aligned} \nabla^2 p(x,y,z) + k^2 p(x,y,z) &= 0, \qquad (x,y,z) \in \Omega \\ \nabla p(x,y,z) + \mathrm{i}\omega\rho \mathbf{v}(x,y,z) &= \mathbf{0}, \qquad (x,y,z) \in \Omega \end{aligned}$$

$$v_n(x,y,z) = 0, \qquad (x,y,z) \in \Sigma_0$$

- Coupling

$$\begin{aligned} g(x,y) = p(x,y,0) + f(x,y), &\qquad (x,y) \in \Sigma \\ \mathrm{i}\omega u(x,y) = v_n(x,y,0), &\qquad (x,y) \in \Sigma \end{aligned}$$

Coupled Problem Reformulated

- Membrane coupled to cavity

$$-T\nabla^2 u(x,y) - \omega^2 \sigma u(x,y) = p(x,y,0) + f(x,y), \qquad (x,y) \in \Sigma$$

$$u(x,y) = 0, \qquad (x,y) \in \Gamma$$

- Cavity coupled to membrane

$$\begin{aligned} \nabla^2 p(x,y,z) + k^2 p(x,y,z) &= 0, \qquad (x,y,z) \in \Omega \\ \nabla p(x,y,z) + \mathrm{i}\omega\rho \mathbf{v}(x,y,z) &= \mathbf{0}, \qquad (x,y,z) \in \Omega \end{aligned}$$

$$v_n(x,y,z) = \begin{cases} 0 & (x,y,z) \in \Sigma_0 \\ \mathrm{i}\omega u(x,y) & (x,y,z) \in \Sigma \end{cases}$$

Weak Form of Coupled Problem

- Membrane coupled to cavity

$$T\int_\Sigma \nabla\psi^{\mathrm{m}}\nabla u \mathrm{d}\Sigma - \omega^2\sigma\int_\Sigma \psi^{\mathrm{m}} u \mathrm{d}\Sigma = \int_\Sigma \psi^{\mathrm{m}} p \mathrm{d}\Sigma + \int_\Sigma \psi^{\mathrm{m}} f \mathrm{d}\Sigma$$

- Cavity coupled to membrane

$$\rho c^2\int_\Omega \nabla\psi^{\mathrm{a}}\nabla p \mathrm{d}\Omega - \omega^2\rho\int_\Omega \psi^{\mathrm{a}} p \mathrm{d}\Omega = \omega^2 z_0^2\int_\Sigma \psi^{\mathrm{a}} u \mathrm{d}\Sigma$$

Discrete Function Spaces

- Membrane coupled to cavity

$$T \int_\Sigma \nabla \mathbf{N}_u^{\mathrm{T}} \nabla \mathbf{N}_u \mathrm{d}\Sigma \, \mathbf{u} - \omega^2 \sigma \int_\Sigma \mathbf{N}_u^{\mathrm{T}} \mathbf{N}_u \mathrm{d}\Sigma \, \mathbf{u} = \underbrace{\int_\Sigma \mathbf{N}_u^{\mathrm{T}} \mathbf{N}_p \mathrm{d}\Sigma}_{\mathbf{A}_{\mathrm{ma}}} \, \mathbf{p} + \int_\Sigma \mathbf{N}_u^{\mathrm{T}} \mathbf{N}_f \mathrm{d}\Sigma \, \mathbf{f}$$

- Cavity coupled to membrane

$$\rho c^2 \int_\Omega \nabla \mathbf{N}_p^{\mathrm{T}} \nabla \mathbf{N}_p \mathrm{d}\Omega \, \mathbf{p} - \omega^2 \sigma \int_\Omega \mathbf{N}_p^{\mathrm{T}} \mathbf{N}_p \mathrm{d}\Omega \, \mathbf{p} = \omega^2 z_0^2 \underbrace{\int_\Sigma \mathbf{N}_p^{\mathrm{T}} \mathbf{N}_u \mathrm{d}\Sigma}_{\mathbf{A}_{\mathrm{am}}} \, \mathbf{u}$$

System Matrices

- Coupled system

$$\mathbf{K}_{\mathrm{m}}\mathbf{u} - \omega^2\mathbf{M}_{\mathrm{m}}\mathbf{u} - \mathbf{A}_{\mathrm{ma}}\mathbf{p} = \mathbf{A}_{\mathrm{m}}\mathbf{f}$$
$$\mathbf{K}_{\mathrm{a}}\mathbf{p} - \omega^2\mathbf{M}_{\mathrm{a}}\mathbf{p} - \omega^2 z_0^2\mathbf{A}_{\mathrm{am}}\mathbf{u} = \mathbf{0}$$

- In matrix form (large, sparse, nonsymmetric)

$$\left(\begin{bmatrix}\mathbf{K}_{\mathrm{m}} & -\mathbf{A}_{\mathrm{ma}} \\ \mathbf{0} & \mathbf{K}_{\mathrm{a}}\end{bmatrix} - \omega^2\begin{bmatrix}\mathbf{M}_{\mathrm{m}} & \mathbf{0} \\ z_0^2\mathbf{A}_{\mathrm{am}} & \mathbf{M}_{\mathrm{a}}\end{bmatrix}\right)\begin{Bmatrix}\mathbf{u} \\ \mathbf{p}\end{Bmatrix} = \begin{Bmatrix}\mathbf{A}_{\mathrm{m}}\mathbf{f} \\ \mathbf{0}\end{Bmatrix}$$

Modal Solution

The modes of the coupled system are not orthogonal, and the eigenvalues are not positive.
We use the normal modes of the uncoupled systems as a base, instead.

- Modal decomposition separately for the mechanical and acoustical systems

$$\begin{Bmatrix} \mathbf{u} \\ \mathbf{p} \end{Bmatrix} = \begin{bmatrix} \boldsymbol{\Psi}_\mathrm{m} & \\ & \boldsymbol{\Psi}_\mathrm{a} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_\mathrm{m} \\ \mathbf{q}_\mathrm{a} \end{Bmatrix}$$

- Modal solution (full, nonsymmetric, but small system)

$$\begin{bmatrix} \boldsymbol{\Lambda}_\mathrm{m} - \omega^2 \mathbf{I} & -\boldsymbol{\Psi}_\mathrm{m}^\mathrm{T} \mathbf{A}_\mathrm{ma} \boldsymbol{\Psi}_\mathrm{a} \\ -\omega^2 z_0^2 \boldsymbol{\Psi}_\mathrm{a}^\mathrm{T} \mathbf{A}_\mathrm{am} \boldsymbol{\Psi}_\mathrm{m} & \boldsymbol{\Lambda}_\mathrm{a} - \omega^2 \mathbf{I} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_\mathrm{m} \\ \mathbf{q}_\mathrm{a} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{\Psi}_\mathrm{m}^\mathrm{T} \mathbf{A}_\mathrm{m} \mathbf{f} \\ \mathbf{0} \end{Bmatrix}$$