The Boundary Element Method in Acoustics

Simulation Methods in Acoustics

Motivation

- Construction of the BEM for the Helmholtz equation
- Components of the Boundary Element Method (BEM)
  1. We have a BVP (= PDE + BC)
  2. We construct the boundary integral representation (BIR) using the fundamental solution of the PDE
  3. BIR is applied for the boundary points to get a boundary integral equation (BIE)
  4. BIE is discretized to get a linear system of algebraic equations. The discretization of the BIE is called the BEM.
  5. The discretized system is solved to get the unknown quantities on the boundary.
  6. Finally, the BIR can be utlizied to compute the radiated quantities (i.e., the sound pressure) in any point of the domain.

Constructing the Green's functions

- Free field Green's functions[1] for the Helmholtz equation

$$\nabla^2 G(\mathbf{x}, \mathbf{x}_0) + k^2 G(\mathbf{x}, \mathbf{x}_0) = -\delta(\mathbf{x} - \mathbf{x}_0) \qquad \mathbf{x} \in \mathbb{R}^d$$

- In general, we can construct the Green's functions using Fourier transform
  - The Fourier transform of the Dirac delta is simple
  - The Laplace operator ∇^2 is transformed to a product in the wave number domain
  - Then, the Green's functions are obtained using inverse Fourier transform
- We have the general expression with setting $\mathbf{x}_0 = \mathbf{0}$

$$G_d(\mathbf{x}) = \mathcal{F}^{-1}\left\{\frac{1}{|\mathbf{k}|^2 - k^2}\right\} \qquad \text{with} \quad \mathbf{k} = (k_x, k_y, k_z)$$

- The inverse transform can be evaluated "mechanically"

[1] a.k.a. fundamental solutions

Reminder on Fourier transforms

- Time domain:

$$U(\omega) = \int_{-\infty}^{\infty} u(t)\mathrm{e}^{-\mathrm{j}\omega t}\,\mathrm{d}t \quad \leftrightarrow \quad u(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(\omega)\mathrm{e}^{\mathrm{j}\omega t}\,\mathrm{d}\omega$$

- Space domain:

$$U(k_x) = \int_{-\infty}^{\infty} u(x)\mathrm{e}^{\mathrm{j}k_x x}\,\mathrm{d}x \quad \leftrightarrow \quad u(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x)\mathrm{e}^{-\mathrm{j}k_x x}\,\mathrm{d}k_x$$

- Time–space $\leftrightarrow$ frequency–wave number

$$U(k_x,\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x,t)\mathrm{e}^{\mathrm{j}k_x x}\mathrm{e}^{-\mathrm{j}\omega t}\,\mathrm{d}x\mathrm{d}t$$

$$u(x,t) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U(k_x,\omega)\mathrm{e}^{-\mathrm{j}k_x x}\mathrm{e}^{\mathrm{j}\omega t}\,\mathrm{d}k_x\mathrm{d}\omega$$

- Using $k_x = \omega/c$ (in 1D) we see that $\mathcal{F}$ is a *plane wave decomposition* and $\mathcal{F}^{-1}$ is a *plane wave superposition*

Green's function in 1D

- Use the general formula in 1D

$$G_1(x) = \mathcal{F}^{-1}\left\{\frac{1}{k_x^2 - k^2}\right\} = \mathcal{F}^{-1}\left\{\frac{1}{(k_x - k)(k_x + k)}\right\} =$$
$$= \frac{1}{2k}\mathcal{F}^{-1}\left\{\frac{1}{k_x - k} - \frac{1}{k_x + k}\right\} = -\frac{1}{2k}\sin\left(k\left|x\right|\right)$$

- We can use that $u(x) = \cos(kx) = \cos(k|x|)$ satisfies

$$\frac{\partial^2 u(x)}{\partial x^2} + k^2 u(x) = 0$$

- Thus, $G_1(x) + c_1 \cos(k|x|)$ is also a free field Green's function
- Let $c_1 = 1/2k\mathrm{j}$

$$G_1^*(x) = \frac{1}{2k}\left(-\frac{\cancel{\mathrm{e}^{\mathrm{j}k|x|}} - \mathrm{e}^{-\mathrm{j}k|x|}}{2\mathrm{j}} + \frac{\cancel{\mathrm{e}^{\mathrm{j}k|x|}} + \mathrm{e}^{-\mathrm{j}k|x|}}{2\mathrm{j}}\right) = \frac{1}{2k\mathrm{j}}\mathrm{e}^{-\mathrm{j}k|x|}$$

Green's function in 2D

- Use the general formula in 2D and use integral

$$G_2(\mathbf{x}) = \mathcal{F}^{-1}\left\{\frac{1}{k_x^2 + k_y^2 - k^2}\right\} = \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{j}\mathbf{k}\cdot\mathbf{x}}}{|\mathbf{k}|^2 - k^2}\,\mathrm{d}\mathbf{k}$$

- Transform to polar coordinates
  - $\mathbf{x} = (x, y) = (r\cos\vartheta, r\sin\vartheta)$
  - $\mathbf{k} = (k_x, k_y) = (\rho\cos\varphi, \rho\sin\varphi)$
  - $\mathbf{k}\cdot\mathbf{x} = r\rho\cos(\vartheta - \varphi)$

$$G_2(r,\vartheta) = \frac{1}{4\pi^2}\lim_{R\to\infty}\int_0^R\int_0^{2\pi}\frac{\rho\mathrm{e}^{-\mathrm{j}\mathbf{k}\cdot\mathbf{x}}}{\rho^2 - k^2}\,\mathrm{d}\varphi\,\mathrm{d}\rho =$$

$$= \frac{1}{4\pi^2}\lim_{R\to\infty}\int_0^R\frac{\rho}{\rho^2 - k^2}\underbrace{\int_0^{2\pi}\mathrm{e}^{-\mathrm{j}r\rho\cos(\vartheta-\varphi)}\,\mathrm{d}\varphi}_{2\pi J_0(r\rho)}\,\mathrm{d}\rho$$

- Where J_0 is the Bessel function (of first kind), order 0

- . . . continued . . .

$$G_2(r,\vartheta) = \frac{1}{2\pi} \underbrace{\lim_{R\to\infty} \int_0^R \frac{\rho J_0(r\rho)}{\rho^2 - k^2}}_{K_0(-\mathrm{j}kr)} \,\mathrm{d}\rho = \frac{1}{2\pi} K_0(-\mathrm{j}kr) =$$

$$= \frac{\mathrm{j}}{4} H_0^{(1)}(kr) = -\frac{\mathrm{j}}{4} H_0^{(2)}(kr)$$

- The new functions are:
  - K_0 – modified Bessel function of second kind, order 0
  - $H_0^{(n)}$ – Hankel function[2] of type n, order 0
  - Smooth (except for $kr \to 0$), complex valued functions, can also be evaluated for complex arguments

[2] a.k.a. Bessel function of third kind

Green's function in 3D

- Following similar procedures as above we get that

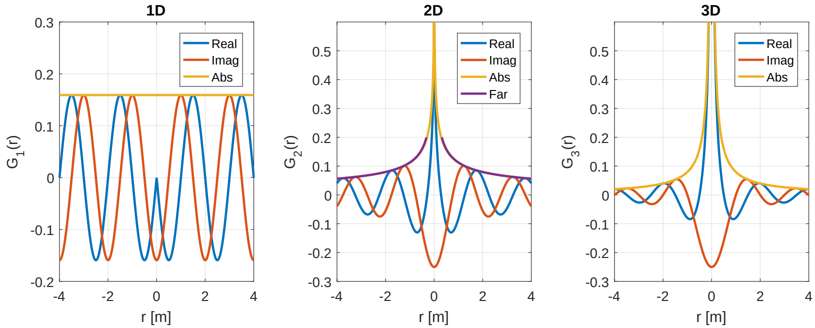
$$G_3(\mathbf{x}, \mathbf{x}_0) = \frac{\mathrm{e}^{-\mathrm{j}kr}}{4\pi r} \qquad \text{with} \quad r = |\mathbf{x} - \mathbf{x}_0|$$

- Finally, we have the free field Green's functions as

$$\begin{aligned} \text{1D:} \qquad & G_1(x, x_0) = \frac{1}{2k\mathrm{j}}\mathrm{e}^{-\mathrm{j}kr} \\ \text{2D:} \qquad & G_2(\mathbf{x}, \mathbf{x}_0) = -\frac{\mathrm{j}}{4}H_0^{(2)}(kr) \\ \text{3D:} \qquad & G_3(\mathbf{x}, \mathbf{x}_0) = \frac{\mathrm{e}^{-\mathrm{j}kr}}{4\pi r} \end{aligned}$$

- Note: in the limit $k \to 0$ we get the free field Green's functions of the Laplace equation

Plots of the Green's functions



- Common properties
  - Oscillation with period $\lambda = 2\pi/k$
  - Derivative of real part discontinuous at $r = 0$
  - Imaginary part smooth in the whole domain
  - Decay $\propto r^{-(d-1)/2}$ (d number of dimensions)
  - In 2D and 3D the functions are singular at $r = 0$

BIE for the Helmholtz equation

- A.k.a. Kirchhoff–Helmholtz integral equation (KHIE)
- We have the inhomogeneous Helmholtz equation as

$$\underbrace{\nabla^2 p(\mathbf{x}) + k^2 p(\mathbf{x})}_{\mathcal{H}\{p(\mathbf{x})\}} = -g(\mathbf{x}) \qquad \mathbf{x} \in \Omega \subseteq \mathbb{R}^d$$

1. Testing

$$\int_\Omega \psi(\mathbf{x}) \left[\nabla^2 p(\mathbf{x}) + k^2 p(\mathbf{x})\right] \, \mathrm{d}\mathbf{x} = \int_\Omega -\psi(\mathbf{x}) g(\mathbf{x})$$

2. Integration by parts (twice)

$$\psi\nabla^2 p = \nabla\cdot(\psi\nabla p) - \nabla\psi\cdot\nabla p = \nabla\cdot(\psi\nabla p) - \nabla\cdot(\nabla\psi p) + \nabla^2\psi p$$

Use this in the formula

$$\int_\Omega \psi\nabla^2 p + \int_\Omega \psi k^2 p = \int_\Omega -\psi g$$

3. Result of integration by parts is

$$\int_\Omega \nabla\cdot(\psi\nabla p) - \int_\Omega \nabla\cdot(\nabla\psi p) + \int_\Omega \nabla^2\psi p + \int_\Omega \psi k^2 p = \int_\Omega -\psi g$$

4. Apply Gauss theorem on the first two integrals

$$\int_\Gamma \psi\frac{\partial p}{\partial n} - \int_\Gamma \frac{\partial G}{\partial n}p + \int_\Omega \underbrace{\left[\nabla^2\psi + k^2\psi\right]}_{\mathcal{H}\{\psi(\mathbf{x})\}} p = \int_\Omega -\psi g$$

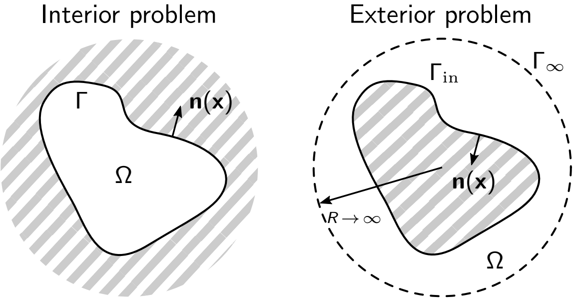
   Notice that the Helmholtz operator $\mathcal{H}$ acts on the test function $\psi(\mathbf{x})$. We exploit this property in the next step.

5. Apply free field Green's function as $\psi(\mathbf{x}) = G(\mathbf{x}-\mathbf{x}_0)$

$$\int_\Gamma G\frac{\partial p}{\partial n} - \int_\Gamma \frac{\partial G}{\partial n}p - \alpha(\mathbf{x}_0)p(\mathbf{x}_0) = \int_\Omega -Gg$$

   with $\alpha(\mathbf{x}_0) = 1,\ 1/2$, or 0 (in Ω, Γ, or otherwise)

The Sommerfeld radiation condition



- Exterior problem: the boundary is composed of the finite boundary and the infinitely far boundary: $\Gamma = \Gamma_{\mathrm{in}} \cup \Gamma_\infty$
- Sommerfeld's condition:
  - Mathematical statement: the boundary integral on Γ_∞ must vanish in free field conditions, i.e.:

$$\int_{\Gamma_\infty} \left(G \frac{\partial p}{\partial n} - \frac{\partial G}{\partial n} p \right) = 0$$

  - Physical meaning: no energy is reflected back from infinity

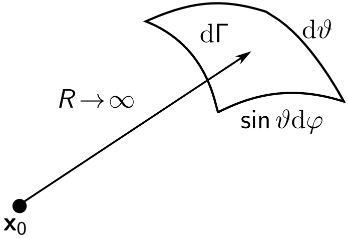
Sommerfeld condition in 3D

- For example, in 3D the integral surely vanishes on Γ_∞

$$\lim_{R\to\infty} \int_\Gamma \left(\frac{\mathrm{e}^{-\mathrm{j}kr}}{4\pi r} \frac{\partial p}{\partial r} - \frac{\partial}{\partial r} \left(\frac{\mathrm{e}^{-\mathrm{j}kr}}{4\pi r} \right) p \right) \mathrm{d}\Gamma = 0$$

- If it vanishes on all small patches $\mathrm{d}\Gamma$
  (Note: G and p are constant on the small patch)

$$\lim_{R\to\infty} R^2 \left(\frac{\mathrm{e}^{-\mathrm{j}kR}}{4\pi R} \frac{\partial p}{\partial r} + (\cancel{1} + \mathrm{j}kR) \frac{\mathrm{e}^{-\mathrm{j}kR}}{4\pi R^2} p \right) \sin\vartheta \mathrm{d}\varphi \mathrm{d}\vartheta = 0$$



- Drop the constants $\sin\vartheta$, $\mathrm{e}^{-\mathrm{j}kR}$, 4π, $\mathrm{d}\vartheta$, $\mathrm{d}\varphi$ to get

$$\lim_{R\to\infty} R\left[\frac{\partial p}{\partial r} + \mathrm{j}kp\right] = 0$$

- Using the Euler equation $\frac{\partial p}{\partial r} = -\mathrm{j}\omega\rho_0 v_r$ we get

$$\lim_{R\to\infty} R\left[p - z_0 v_r\right] = 0$$

- Similarly, in d dimensions we have

$$\lim_{R\to\infty} R^{\frac{d-1}{2}}\left[p - z_0 v_r\right] = 0$$

- We can verify that the free field Green's functions all satisfy the Sommerfeld condition ($p = G_1$, G_2, or G_3 above)
- This means that for any radiator radiating finite energy the boundary integrals on Γ_∞ can be omitted

Weak form of the Kirchhoff–Helmholtz integral equations

- The Kirchoff–Helmholtz integral equation (KHIE) is a continuous equation, discretisation is needed to solve it
- Write the KHIE for a boundary point: $\mathbf{x}_0 \in \Gamma$
- First, we construct the weak form of the integral equation by multiplying by the test function $\phi(\mathbf{x}_0)$ and integrating over the surface Γ
- Assume zero incident field for the time being and denote $\partial p(\mathbf{x})/\partial n(\mathbf{x})$ as $q(\mathbf{x})$ to get

$$\int_\Gamma \phi(\mathbf{x}_0) \int_\Gamma G(\mathbf{x}-\mathbf{x}_0) q(\mathbf{x}) \,\mathrm{d}\mathbf{x} \,\mathrm{d}\mathbf{x}_0 -$$
$$\int_\Gamma \phi(\mathbf{x}_0) \int_\Gamma \frac{\partial G(\mathbf{x}-\mathbf{x}_0)}{\partial n} p(\mathbf{x}) \,\mathrm{d}\mathbf{x} \,\mathrm{d}\mathbf{x}_0 = \int_\Gamma \phi(\mathbf{x}_0) \frac{1}{2} p(\mathbf{x}_0) \,\mathrm{d}\mathbf{x}_0$$

Discretization

- We discretize the boundary into boundary elements

$$\Gamma \approx \bigcup_{i=1}^{E} \Gamma_i \qquad \text{with} \quad \Gamma_i \cap \Gamma_j = \emptyset \quad \text{if} \quad i \neq j$$

- Discretization of the boundary data by shape functions

$$p(\mathbf{x}) = \sum_j N_j^{(p)}(\mathbf{x}) p_j \qquad \mathbf{x} \in \Gamma$$

$$q(\mathbf{x}) = \frac{\partial p(\mathbf{x})}{\partial n} = \sum_j N_j^{(q)}(\mathbf{x}) q_j \qquad \mathbf{x} \in \Gamma$$

$$\phi(\mathbf{x}) = \sum_i N_i^{(\phi)}(\mathbf{x}) w_i \qquad \mathbf{x} \in \Gamma$$

- Note: in the FEM, the derivatives of the shape and test functions appeared $\rightarrow$ at least piecewise linear shape functions and the Galerkin method were needed
- In the BEM we don't have derivatives and have much more freedom in choosing shape and test functions

The collocation form

- Shape and test function choices:
  - $N_j^{(p)}$, $N_j^{(q)}$: piecewise constant[3] over the j-th element
  - $\phi_i(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x_i})$: Dirac-delta at the center of the i-th element
- By the sifting property of the Dirac-delta, double integrals boil down to *single integrals*

$$\sum_j \underbrace{\int_\Gamma G(\mathbf{x} - \mathbf{x}_i) N_j^{(q)}(\mathbf{x}) \, \mathrm{d}\mathbf{x}}_{G_{ij}} \, q_j - \sum_j \underbrace{\int_\Gamma G_n(\mathbf{x} - \mathbf{x}_i) N_j^{(p)}(\mathbf{x}) \, \mathrm{d}\mathbf{x}}_{H_{ij}} \, p_j = \frac{1}{2} p_i$$

- The shape functions are non-zero only over one (or a few) element(s), thus, integration is carried out element-by-element
- Matrix form:

$$\mathbf{Gq} - \mathbf{Hp} = \frac{1}{2}\mathbf{p}$$

- Solution for $\mathbf{p}$ (scattered pressure on surface)

$$\mathbf{p} = \left(\mathbf{H} + \frac{1}{2}\mathbf{I}\right)^{-1} \mathbf{Gq}$$

[3] This is the simplest choice, other choices also possible

Galerkin form

- Shape and test function choices:
  - $N_j^{(p)}$, $N_j^{(q)}$: e.g. piecewise *polynomial* (constant, linear, quadratic, etc.), 1 in the i-th node, 0 in other nodes
  - Same as the shape functions: $\phi_i^{(p,q)}(\mathbf{x}) = N_i^{(p,q)}(\mathbf{x})$
- Double integrals (carried out element pair-by-element pair)

$$\sum_j \underbrace{\iint_\Gamma N_i^{(q)}(\mathbf{x}_0) G(\mathbf{x}-\mathbf{x}_0) N_j^{(q)}(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{x}_0}_{G_{ij}}\ q_j -$$

$$\sum_j \underbrace{\iint_\Gamma N_i^{(p)}(\mathbf{x}_0) G_n(\mathbf{x}-\mathbf{x}_0) N_j^{(p)}(\mathbf{x})\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{x}_0}_{H_{ij}}\ p_j = \frac{1}{2} \underbrace{\int_\Gamma N_i^{(p)}(\mathbf{x}_0) N_j^{(p)}(\mathbf{x_0})\,\mathrm{d}\mathbf{x}_0}_{B_{ij}}\ p_j$$

- Matrix form (**B** – boundary mass matrix, sparse, real valued):

$$\mathbf{Gq} - \mathbf{Hp} = \frac{1}{2}\mathbf{Bp}$$

- Solution for **p** (scattered pressure on surface)

$$\mathbf{p} = \left(\mathbf{H} + \frac{1}{2}\mathbf{B}\right)^{-1} \mathbf{Gq}$$

Surface system matrix properties

- Common properties of matrices **G** and **H** (both for collocation and Galerkin formalism)
  1. Fully populated
  2. Complex valued
  3. *Frequency* (wave number k) dependent
  4. Contain singular integrals over elements
- Collocation: matrices asymetric in general
- Galerkin: matrices are symmetric
- Size depends on shape function choices
  - Elementwise constant shape functions: $E \times E$
  - Elementwise linear shape functions: $N \times N$

Computing the radiated field

- For the computation of the radiated field, the boundary integral representation (BIR) is used.

$$\int_\Gamma G(\mathbf{x},\mathbf{x}_0)q(\mathbf{x})\,\mathrm{d}\mathbf{x} - \int_\Gamma G'_n(\mathbf{x},\mathbf{x}_0)p(\mathbf{x})\,\mathrm{d}\mathbf{x} = p(\mathbf{x}_0) \qquad \mathbf{x}_0 \in \Omega$$

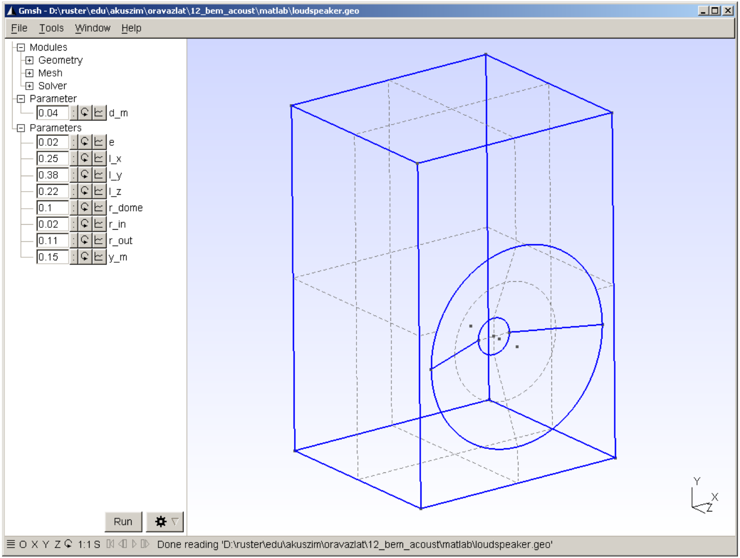
  Notice, that this is a simple forward step, as we already know the surface quantities
- As we have discretized the surface variables, the integrals on the l.h.s. can be written as matrix–vector products.
- If we choose a number of field points, we get the field point pressures $\mathbf{p}_f$ by a simple multiplication

$$\mathbf{G}_f\mathbf{q}_s - \mathbf{H}_f\mathbf{p}_s = \mathbf{p}_f$$

- $\mathbf{G}_f$ and $\mathbf{H}_f$ are also full, frequency dependent, but contain no singular integrals

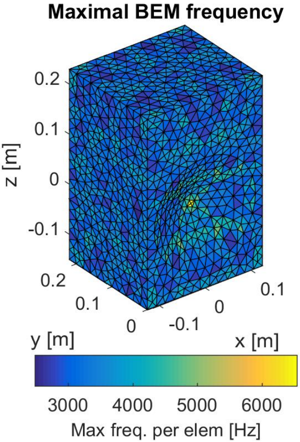
Example – A radiation problem I.

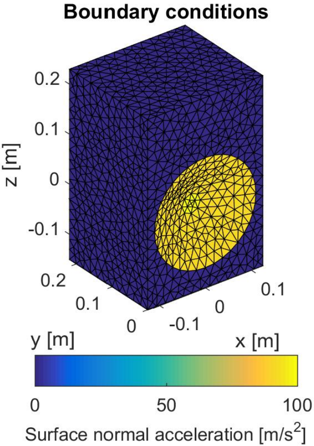
- Example problem: exterior radiation from a loudspeaker



Example – A radiation problem II.

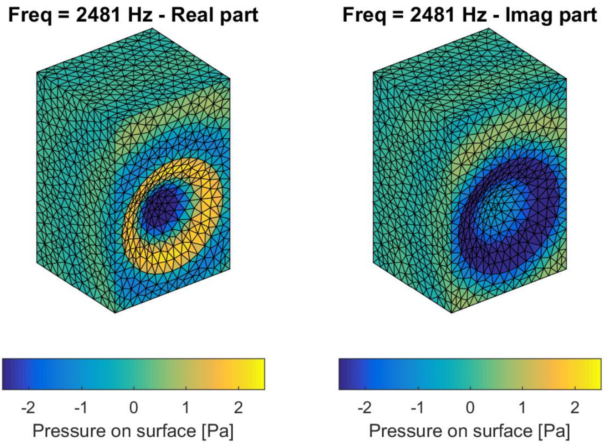
- Computation at different frequencies, maximum frequency is limited by the largest elements (rule of thumb: $l_e < \lambda/6$)
- Constant acceleration on the membrane is assumed
- The membrane is not planar $\rightarrow$ normal velocity is not constant





Example – A radiation problem III.

- Solution process (for each frequency)
  1. Assembe matrices $\mathbf{G}$, $\mathbf{H}$, $\mathbf{G}_f$, $\mathbf{H}_f$
  2. Compute surface pressure by solving the KHIE
  3. Compute field point pressures by using the BIR



Example – A radiation problem IV.

- Field point result – Directivity of the loudspeaker
- Frequency dependency is clearly observed
  - Low frequencies – radiation is nearly spherical
  - Higher frequencies – focused radiation, side lobes appear
  - Vertical directivity is asymmetric, as expected