Numerical Integration

Simulation Methods in Acoustics

Motivation

Integration of a BEM matrix element

$$G_{ij} = \sum_e \int_{\Gamma_e} G(\mathbf{x} - \mathbf{x}_i) N_j(\mathbf{x}) \mathrm{d}\mathbf{x} \tag{1}$$

$$= \sum_e \int_{\Xi} G(\mathbf{x}^e(\boldsymbol{\xi}) - \mathbf{x}_i) N_j(\boldsymbol{\xi}) J^e(\boldsymbol{\xi}) \mathrm{d}\boldsymbol{\xi} \tag{2}$$

where

$$\mathbf{x}^e(\boldsymbol{\xi}) = \sum_k \mathbf{x}_k^e L_k(\boldsymbol{\xi}) \tag{3}$$

$$J^e(\xi, \eta) = \left| \frac{\partial \mathbf{x}^e}{\partial \xi} \times \frac{\partial \mathbf{x}^e}{\partial \eta} \right| \tag{4}$$

Conclusion: Numerical integration of complex functions over regular (standard) domains

Problem Definition

Quadrature rule

$$\int_{-1}^{1} f(x)\mathrm{d}x \approx \sum_{j=1}^{n} f(x_j)w_j \tag{5}$$

Classification:

- Newton-Cotes quadrature (equidistant interpolation)
- Gaussian quadrature

Quadrature size: n number of function samples
Quadrature order: Highest polynomial order integrated accurately.

Newton-Cotes quadrature

Lagrange interpolation with equidistant samples

$$f(x) \approx \sum_{j=1}^{n} f(x_j) L_j(x), \quad L_j(x) = \prod_{k=1, k \neq j}^{n} \frac{x - x_k}{x_j - x_k} \tag{6}$$

$$\int_{-1}^{1} f(x) \mathrm{d}x \approx \sum_{j=1}^{n} f(x_j) \underbrace{\int_{-1}^{1} L_j(x) \mathrm{d}x}_{w_j} \tag{7}$$

Example: Simpson rule $n = 3$, $x_j = [-1, 0, 1]$, $w_j = \left[\frac{1}{3}, \frac{4}{3}, \frac{1}{3}\right]$

Gaussian Quadrature

Let $f(x)$ be a polynomial of order $2n-1$

$$f(x) = p_{2n-1}(x) \tag{8}$$

Divide p by the n-th order polynomial q

$$p_{2n-1}(x) = q_n(x)d_{n-1}(x) + r_{n-1}(x) \tag{9}$$

Then the integral yields

$$\int_{-1}^{1} p_{2n-1}(x)\mathrm{d}x = \int_{-1}^{1} q_n(x)d_{n-1}(x)\mathrm{d}x + \int_{-1}^{1} r_{n-1}(x)\mathrm{d}x \tag{10}$$

If q is orthogonal to every polynomial up to order $n-1$ then the integral simplifies to the integral of the remainder

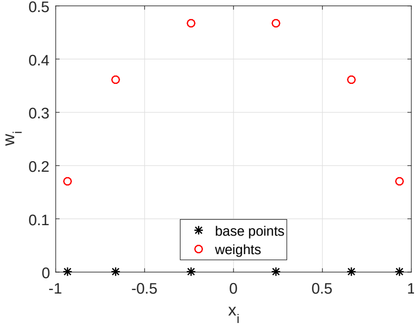
$$\int_{-1}^{1} p_{2n-1}(x)\mathrm{d}x = \int_{-1}^{1} r_{n-1}(x)\mathrm{d}x \tag{11}$$

Gaussian Quadrature

Let x_j be the n roots of $q_n(x)$. In this case

$$p_{2n-1}(x_j) = r_{n-1}(x_j) \tag{12}$$

so we have n samples of the $n-1$-th order remainder. The remainder is then reconstructed and integrated.



n	q	x_j	w_j
1	x	0	2
2	$3x^2-1$	$\pm 1/\sqrt{3}$	1
3	$5x^3-3x$	$0, \pm\sqrt{3/5}$	8/9, 5/9

Integration over rectangles

Tensor product quadrature

$$\int_{-1}^{1}\int_{-1}^{1} f(x,y)\mathrm{d}x\mathrm{d}y = \sum_i \left(\sum_j f(x_i, y_j) w_j \right) w_i \qquad (13)$$

Quadrature points: (x_i, x_j), weights: $w_i \cdot w_j$

Integration over triangles

$$\int_0^1 \int_0^x f(x,y)\mathrm{d}y\mathrm{d}x \tag{14}$$

Duffy transform: $y = \eta x$, $\mathrm{d}y = x\mathrm{d}\eta$

$$\int_0^1 \int_0^1 f(x, x\eta)x\mathrm{d}\eta\mathrm{d}x = \sum_i \sum_j f(x_i, x_i\eta_j)x_i w_i w_j \tag{15}$$

Equivalent to integration over a distorted rectangle with corners $(0,0),(1,0),(0,1),(0,0)$

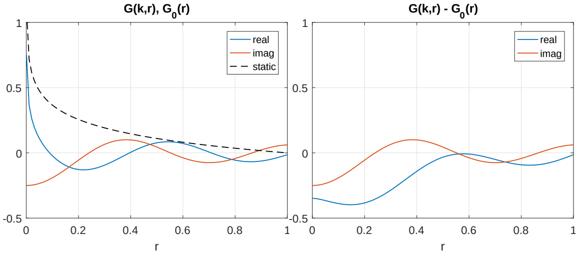
Singular integrals – Static Part Subtraction

$$G_3(k,r) = \frac{\mathrm{e}^{-\mathrm{i}kr}}{4\pi r}, \qquad G_3(0,r) = \frac{1}{4\pi r} \tag{16}$$

Static part subtraction:

$$G_3(k,r) = G_3(0,r) + \underbrace{\left(G_3(k,r) - G_3(0,r)\right)}_{\text{regular, Gaussian quad}} \tag{17}$$

$$G_3(k,r) - G_3(0,r) = \frac{-\mathrm{i}k}{4\pi} \sum_{n=1}^{\infty} \frac{(-\mathrm{i}kr)^{n-1}}{n!} \tag{18}$$



The static part is integrated analytically