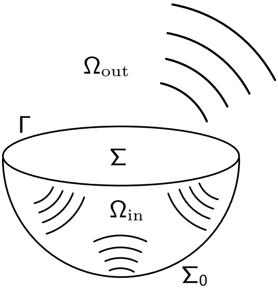
Vibroacoustic FEM–BEM coupling

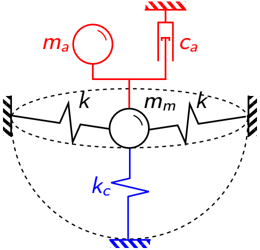
Simulation Methods in Acoustics

FEM – BEM coupling

- Motivation: by FEM/BEM coupling we can solve
  1. Interior acoustic field (FEM) coupled to outer field (BEM)
  2. Structural vibration (FEM) coupled to acoustic field (BEM)
  3. Interior acoustic (FEM) + structure (FEM) coupled to exterior acoustic field (BEM)
- Strategy
  - In the discretization process we use same shape functions for BEM and FEM. This enables matrix level coupling without additional constraint equations.
  - Coupled system as a block matrix equation
- Challenges
  - BEM matrices are full and frequency dependent
  - Modal solution not possible due to frequency dependence
- Demonstrator application: drum
  1. Vibrating membrane + radiation to exterior infinite space
  2. Full model: Vibrating membrane + closed cavity + radiation

Membrane coupled to interior / exterior acoustic fields





- Coupling effects and their low frequency approximations
  1. Cavity (Interior field): increases the stiffness of the membrane
  2. Acoustical radiation (Exterior field): radiation of energy

  (Interior and exterior fields coupled by the membrane only.)
- First, we deal with the exterior field only, then the simultaneous effect of both couplings are examined.

Membrane coupled to exterior acoustic field

- Expectations:
  1. Outer air: additional mass, but no additional stiffness
     Eigenfrequency will be lowered
  2. Radiation of sound $\rightarrow$ membrane is dambed
     Sound radiation is a loss from the viewpoint of the membrane
- Approximation for the first mode:
  - Membrane mass: $m_m = A_m \sigma = R^2 \pi \sigma$
  - Additional air mass (layer on the whole membrane):
    $m_a \approx 0.85 R A_m \rho_0$

$$\frac{\omega'}{\omega} = \frac{\sqrt{k/(m_m + m_a)}}{\sqrt{k/m_m}}$$

  - With $\sigma = 0.27\,\mathrm{kg/m^2}$, $\rho_0 = 1.2\,\mathrm{kg/m^3}$, $R = 0.3\,\mathrm{m}$

$$\frac{\omega'}{\omega} \approx \frac{2}{3}$$

first mode shifted downwards by a pure fifth!

A coupled problem – The drum revised

- Problem definition
  - Mechanical system (Helmholtz equation + BC)

$$-T\nabla^2 u(x,y) - \omega^2 \sigma u(x,y) = f(x,y) - p^{(\mathrm{out})}(x,y,0) \qquad \mathbf{x} \in \Sigma$$
$$u(x,y) = 0 \qquad \mathbf{x} \in \Gamma$$

  - Acoustical system (Helmholtz and Euler equations)

$$\nabla^2 p^{(\mathrm{out})}(\mathbf{x}) - k^2 p^{(\mathrm{out})}(\mathbf{x}) = 0 \qquad \mathbf{x} \in \Omega^{(\mathrm{out})}$$
$$\nabla p^{(\mathrm{out})}(\mathbf{x}) + \mathrm{j}\omega\rho_0 \mathbf{v}(\mathbf{x}) = \mathbf{0} \qquad \mathbf{x} \in \Omega^{(\mathrm{out})}$$

  - BCs for the acoustical system
    Membrane velocity equals the normal particle velocity (in Σ), the body of the drum is acoustically rigid (non-moving)

$$v_n(\mathbf{x}) = \mathrm{j}\omega u(\mathbf{x}) \qquad \mathbf{x} \in \Sigma$$
$$v_n(\mathbf{x}) = 0 \qquad \mathbf{x} \in \Sigma_0$$

Using KHIE for exterior acoustics

- Write the KHIE for $\mathbf{x}_0 \in \partial\Omega = \Sigma \cup \Sigma_0$
- Note: with $\mathbf{n}$ pointing *inwards* the domain Ω

$$\int_{\partial\Omega} \left[G'_n(\mathbf{x}, \mathbf{x}_0) p^{(\mathrm{out})}(\mathbf{x}) - G(\mathbf{x}, \mathbf{x}_0) q^{(\mathrm{out})}(\mathbf{x}) \right] \mathrm{d}\mathbf{x} = \frac{1}{2} p^{(\mathrm{out})}(\mathbf{x}_0)$$

- Discretize to get the exterior acoustic BEM
  1. Test with the test function $\phi(\mathbf{x}_0)$
  2. Discretize geometry by boundary elements
  3. Discretize surface variables using shape functions
     - Ensure compatibility with the FEM
     - Use same shape functions and Galerkin method
     - Without additional constraints, we can couple matrices
- We obtain an algebraic set of equations as

$$\mathbf{H}\mathbf{p}^{(\mathrm{out})} - \mathbf{G}\mathbf{q}^{(\mathrm{out})} = \frac{1}{2} \mathbf{B}\mathbf{p}^{(\mathrm{out})}$$

Coupling on the matrix level I.

- Matrix equations

$$\mathbf{K}_m\mathbf{u} - \omega^2\mathbf{M}_m\mathbf{u} = \mathbf{A}_m\mathbf{f} - \mathbf{A}_{mo}\mathbf{p}^{(\mathrm{out})} \qquad \text{Mechanical FEM}$$

$$-\mathbf{G}\mathbf{q}^{(\mathrm{out})} + \mathbf{H}\mathbf{p}^{(\mathrm{out})} = \frac{1}{2}\mathbf{B}\mathbf{p}^{(\mathrm{out})} \qquad \text{Acoustical BEM}$$

(Coupling matrix $\mathbf{A}_{mo}$ same as for FEM–FEM coupling.)

- Make use of the Euler equation to relate u and q

$$q = \frac{\partial p}{\partial n} = -\mathrm{j}\omega\rho_0 v_n = \rho_0\omega^2 u$$

- With $\mathbf{S}_m(\omega) = \mathbf{K}_m - \omega^2\mathbf{M}_m$ we have (red: coupling parts)

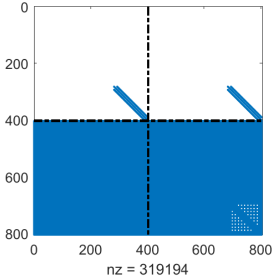
$$\begin{aligned} \mathbf{S}_m(\omega)\mathbf{u} + A_{mo}\mathbf{p}^{(\mathrm{out})} &= \mathbf{A}_m\mathbf{f} \\ -\rho_0\omega^2\mathbf{G}(\omega)\mathbf{u} + \left(\mathbf{H}(\omega) - \frac{1}{2}\mathbf{B}\right)\mathbf{p}^{(\mathrm{out})} &= \mathbf{0} \end{aligned}$$

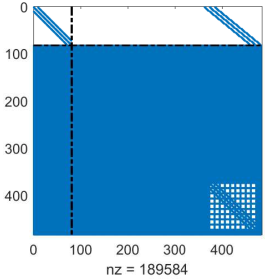
Coupling on the matrix level II.

- Writing it as a block matrix equation

$$\begin{bmatrix} \mathbf{S}_m(\omega) & \mathbf{A}_{mo} \\ -\rho_0\omega^2\mathbf{G}(\omega) & (\mathbf{H}(\omega) - \frac{1}{2}\mathbf{B}) \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ \mathbf{p}^{(\mathrm{out})} \end{Bmatrix} = \begin{Bmatrix} \mathbf{A}_m\mathbf{f} \\ \mathbf{0} \end{Bmatrix}$$

- $\mathbf{S}_m$, $\mathbf{A}_{mo}$ and $\mathbf{A}_m$ are sparse, while $\mathbf{G}$ and $\mathbf{H}$ are full
- Matrix size is reduced by constraints: $\mathbf{u}$ is zero on Σ_0 and Γ
- Matrix structure (notice some zeros in $\mathbf{H}$ on plane elements)



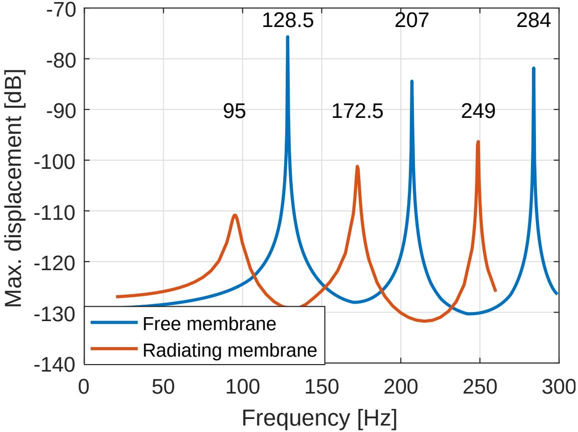


Solution process

- Compute FEM system matrices $\mathbf{K}_m$, $\mathbf{M}_m$, $\mathbf{A}_{mo}$ and $\mathbf{A}_m$
- For each testing frequency ω_i:
  1. Compute BEM matrices $\mathbf{G}(\omega_i)$, $\mathbf{H}(\omega_i) - \frac{1}{2}\mathbf{B}$
  2. Assemble block matrix and right hand side excitation vector
  3. Solve block equation to get surface quantities
  4. Optional: Use BIR to get radiated pressure in field points
- Finding eigenfrequencies:
  - Modal solution not possible (frequency dependent matrices)
  - Find maxima in $\mathbf{u}(\omega)$ or $\mathbf{p}^{(\text{out})}(\omega)$
    (Coarse approx. first, then refining the scale near maxima)

Result – Transfer function

- Free membrane: Eigenfrequencies are the ideal analytical ones, *infinite response* at eigenfrequencies
- Radiating membrane: Eigenfrequencies shifted downwards, *finite response* at eigenfrequencies



Fully coupled membrane

- Assemble the drum: membrane + exterior + interior fields
- We will use mech. FEM + acou. BEM + acou. FEM

1. Membrane

$$-T\nabla^2 u(x,y) - \omega^2\sigma u(x,y) = f(x,y) + p^{(\mathrm{in})}(x,y,0) - p^{(\mathrm{out})}(x,y,0) \qquad \mathbf{x}\in\Sigma$$
$$u(x,y) = 0 \qquad \mathbf{x}\in\Gamma$$

2. Exterior acoustic field using KHIE
Coupling: Use that $q^{(\mathrm{out})} = \rho_0\omega^2 u$ on the membrane ($\mathbf{x}\in\Sigma$)

$$\int_{\Sigma\cup\Sigma_0}\left[G'_n(\mathbf{x},\mathbf{x}_0)p^{(\mathrm{out})}(\mathbf{x}) - G(\mathbf{x},\mathbf{x}_0)q^{(\mathrm{out})}(\mathbf{x})\right]\mathrm{d}\mathbf{x} = \frac{1}{2}p^{(\mathrm{out})}(\mathbf{x}_0) \quad \mathbf{x},\mathbf{x}_0\in\Sigma\cup\Sigma_0$$

3. Interior acoustic field
Coupling: Use that $v_n^{(\mathrm{in})} = \mathrm{j}\omega u$ on the membrane ($\mathbf{x}\in\Sigma$)

$$\nabla^2 p^{(\mathrm{in})}(\mathbf{x}) + k^2 p^{(\mathrm{in})}(\mathbf{x}) = 0$$
$$\frac{\partial p^{(\mathrm{in})}(\mathbf{x})}{\partial n(\mathbf{x})} + \mathrm{j}\omega\rho_0 v_n^{(\mathrm{in})}(\mathbf{x}) = 0$$

Full coupling on the matrix level

- Matrix equations are obtained after discretization using compatible shape functions (as above)

$$\mathbf{K}_m\mathbf{u} - \omega^2\mathbf{M}_m\mathbf{u} = \mathbf{A}_m\mathbf{f} - A_{mo}\mathbf{p}^{(\mathrm{out})} + A_{mc}\mathbf{p}^{(\mathrm{in})} \qquad \text{mech. FEM}$$

$$\mathbf{G}\mathbf{q}^{(\mathrm{out})} - \mathbf{H}\mathbf{p}^{(\mathrm{out})} = \frac{1}{2}\mathbf{B}\mathbf{p}^{(\mathrm{out})} \qquad \text{acou. BEM}$$

$$\mathbf{K}_a\mathbf{p}^{(\mathrm{in})} - \omega^2\mathbf{M}_a\mathbf{p}^{(\mathrm{in})} = -\mathrm{j}\omega\mathbf{A}_{am}\mathbf{v} \qquad \text{acou. FEM}$$

- The coupled system using the membrane velocity (displacement) appearing in all equations

$$\begin{aligned} \mathbf{S}_m(\omega)\mathbf{u} + A_{mo}\mathbf{p}^{(\mathrm{out})} \qquad & -\mathbf{A}_{mc}\mathbf{p}^{(\mathrm{in})} = \mathbf{A}_m\mathbf{f} \\ -\rho_0\omega^2\mathbf{G}(\omega)\mathbf{u} + \left(\mathbf{H}(\omega) - \frac{1}{2}\mathbf{B}\right)\mathbf{p}^{(\mathrm{out})} \qquad & = \mathbf{0} \\ -\omega^2 z_0^2\mathbf{A}_{cm}\mathbf{u} \qquad & +\mathbf{S}_a(\omega)\mathbf{p}^{(\mathrm{in})} = \mathbf{0} \end{aligned}$$

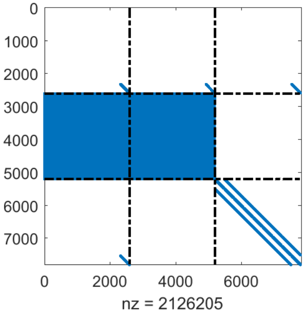
- Notice that the interior and exterior fields are not coupled directly (empty blocks in the system matrix)

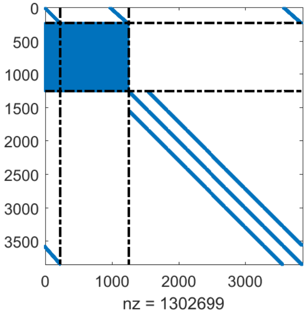
Matrix structure and solution

- Notice that the 3 systems have different number of DOFs

$$N_{\mathrm{memb}} < N_{\mathrm{surf}} < N_{\mathrm{volume}}$$

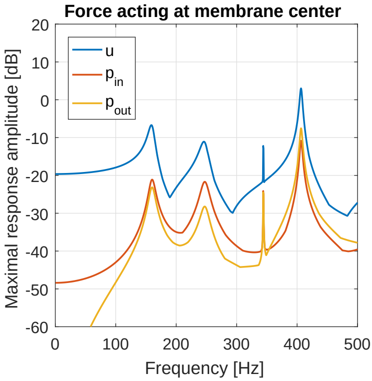
- Solution process is same as above for FEM/BEM coupling
- Left: fully padded matrix, right: reduced matrix

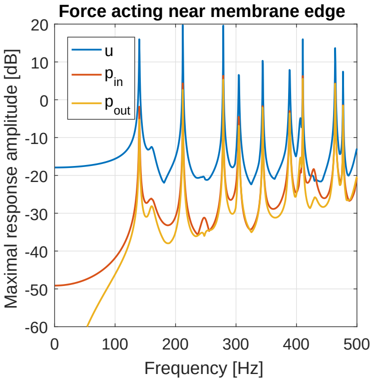




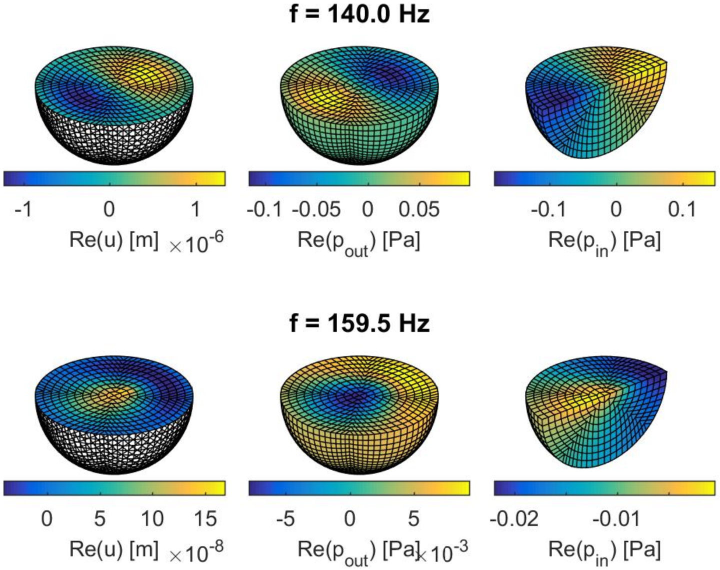
Results – Transfer function

- Solving with multiple r.h.s.: different positions of acting force
- Joint effect of cavity and exterior radiation is observed
- Different modes excited efficiently depending on the position of the force, the modes have significantly different damping

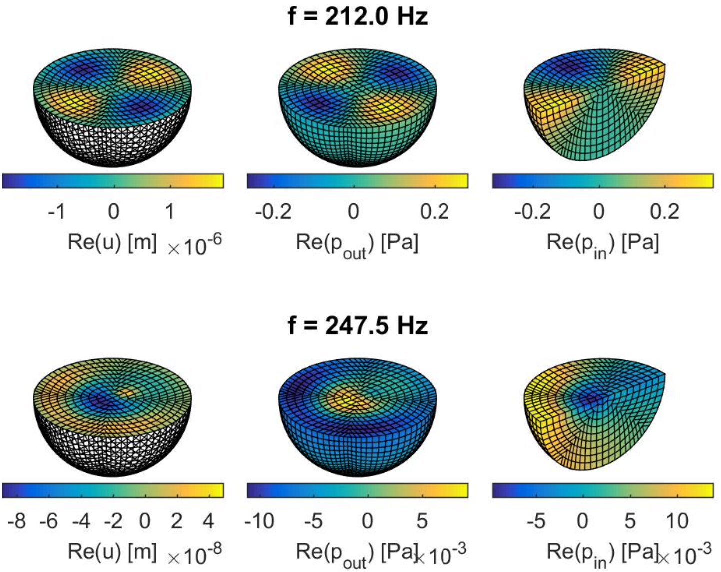




Results – The first few modes I.



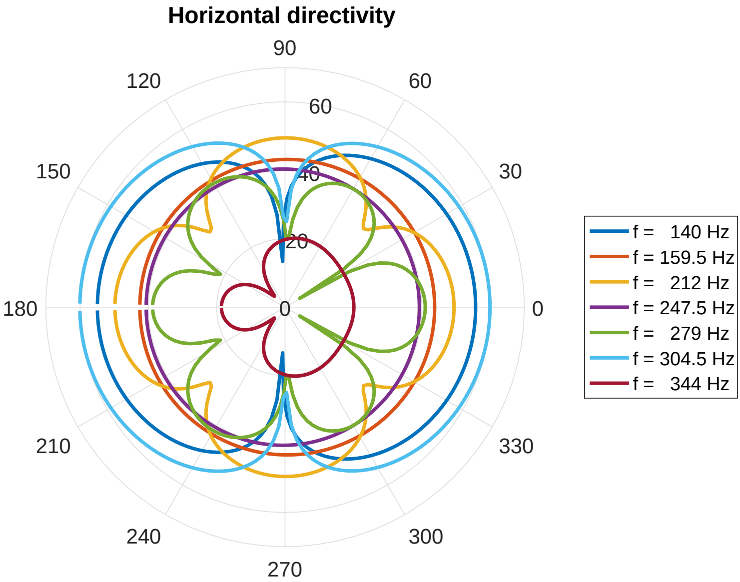
Results – The first few modes II.



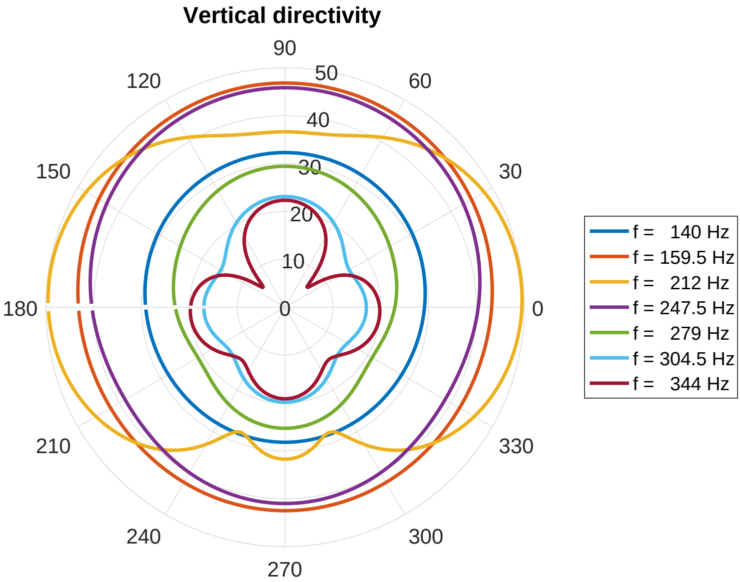
Interpretation of the results

- Notice that the first two modes $(1, 0)$ and $(0, 1)$ became reversed in frequency – the spring effect of the cavity is very large for the $(1, 0)$ mode, smaller for other modes
- For all modes we see that where the membrane displacement is positive, the outer pressure is negative, and inner pressure is positive. This matches our expectations.
- We can also observe some asymmetry due to the position of the force excitation
- It can also be expected that the directivity of the drum is strongly dependent on the frequency. This can be checked by computing the outer pressure $p^{(\mathrm{out})}$ in some field points.

Results – Directivities I.



Results – Directivities II.



Conclusions

- Using FEM/BEM coupling we were able to compute a relatively complex problem
- By compatible shape functions, we can write the coupled system by a matrix level coupling
- We get a frequency dependent block structured matrix equation that is solved at different test frequencies
- We get the (1) structure displacement, (2) inner and (3) outer pressure fields and can also compute the (4) radiated sound field in a number of far field points to get the directivities
- If the solution is computed with a sufficient frequency resolution, IFFT can be used to get time domain sound samples