The General Relation of Wave Field Synthesis and the Spectral Division Method

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***Abstract*—The abstract goes here.**

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I. Introduction

The physical reconstruction of arbitrary sound fields over an extended listening area—generally termed as *sound field synthesis (SFS)*—has been the subject of extensive research over the last three decades [1]. As a common characteristics, different SFS techniques apply a densely spaced loudspeaker ensemble, termed as the *secondary source distribution (SSD)*. The loudspeakers are fed with properly chosen *driving functions*, so that their resultant sound field coincides with the target sound field in the receiving area.

Regarding the methodology how driving functions are derived SFS techniques can be classified into two main areas:

The *explicit solution* aims at the direct solution of the inverse problem constituted by the general SFS integral. Once an orthogonal / spectral decomposition is known for the actual SSD geometry, the driving function spectrum may be obtained by comparison with the corresponding spectral coefficients of the involved sound fields taken on a control surface [2]. Hence, these solutions are often termed as *mode-matching solutions* [3]. For simple geometries the explicit solution are well-known. The solution in spherical and cylindrical SSD surfaces are termed as *Nearfield Compensated Higher Order Ambisonics (NFC HOA)* [4]–[7], giving a wideband extension for traditional Ambisonics technique [8]. Recently the explicit solution for planar and linear geometries was introduced by Ahrens first for a virtual plane wave [9]–[11], later extended to arbitrary virtual source models. The approach is referred to as the *Spectral Division Method (SDM)*. The explicit solution accounts for the global description of the involved sound field, therefore it may be termed as *global solution* [12].

As an alternative, *Wave Field Synthesis (WFS)*—giving an *implicit solution*—is based on the single layer boundary integral formulation of arbitrary sound fields, containing the required driving functions implicitly [13]. All WFS approaches apply the *stationary phase approximation (SPA)* to the boundary integral in order to reduce the problem dimensionality from 3 to 2. *Traditional WFS*, as given by Berkhout et. al [14], [15] utilized the Rayleigh integral representations to obtain driving functions for linear SSDs with dipole characteristics, later extended to monopole secondary sources [16]–[18]. Traditional WFS considered only directive virtual point sources and ensured optimal synthesis on a reference line parallel to the SSD. *Revisited WFS* [19] extended the theory for arbitrary virtual source models, including non-linear SSD curves by applying the Kirchhoff-approximation to the 2.5D Neumann Rayleigh integral and ensured optimal synthesis of 2D virtual fields on a curve containing the pre-defined reference point. In a former article by the present authors a *unified WFS theory* was proposed [20]. It was verified, that within the validity of the SPA the synthesis of an arbitrary virtual source may be referenced on an arbitrary smooth, convex reference curve by defining proper referencing functions. In this unified WFS theory the former approaches occur as special cases. As WFS finds the solution by matching the local wavefronts of the SSD and the virtual sound field, it is often termed as *local solution* [12].

The relation of the implicit and explicit solutions has already been investigated in numerous studies. It was verified, that revisited WFS constitutes a high frequency approximation of Nearfield Compensated Infinite Order Ambisonics in a circular SSD geometry [12]. Furthermore the same connection was shown between WFS and SDM for the special cases of a virtual point source [21], a virtual plane wave [22], [23], and recently for an arbitrary 2D virtual sound field [20].

The present article establishes the link between the Spectral Division Method and unified WFS theory by showing their equivalence within the validity of the stationary phase approximation, regardless of the virtual sound field. The paper briefy recalls the two approaches including the introduction of the local wavenumber concept. Then, using the SPA, general spatial domain SDM driving functions are derived. The driving functions require the evaluation of the target field along an arbitrary, pre-defined reference curve, opposed to WFS, where the driving functions are formulated in terms of the virtual field's normal derivative on the SSD. Finally it is shown, that the derived SDM solution is equivalent to the unified 2.5D WFS driving functions stemming from the 2.5D Neumann Rayleigh integral, and inherently contains the referencing curve concept.

II. Theoretical basics

A. SFS problem formulation

The general SFS geometry, applying a linear SSD is depicted in Figure 1. Assume a continuous linear set of secondary sources, located at $\mathbf{x}_0 = [x_0,\ 0,\ 0]^{\mathrm{T}}$. The listening area is a horizontal half-plane, containing the SSD $\mathbf{x} = [x,\ y >$

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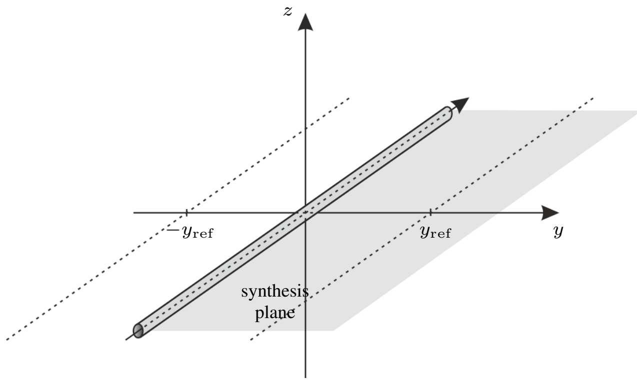

Fig. 1. Geometry for the general SFS problem applying a linear SSD

$0,\ 0]^{\mathrm{T}}$. Assuming a harmonic time dependence given by $\mathrm{e}^{\mathrm{j}\omega t}$ the synthesized field is given by

$$P(\mathbf{x},\omega) = \int_{-\infty}^{\infty} D(x_0,\omega) G(x - x_0, y, \omega) \mathrm{d}x_0, \qquad (1)$$

where $G(\mathbf{x},\omega)$ describes the field of an individual SSD element and $D(x_0,\omega)$ is the driving function to be found, so that the synthesized field equals the target field in the listening area. Generally the SSD elements are regarded to be acoustic point sources, described by the fullspace, free field 3D Green's function

$$G(\mathbf{x},\omega) = \frac{1}{4\pi} \frac{\mathrm{e}^{-\mathrm{j}k|\mathbf{x}|}}{|\mathbf{x}|}, \qquad (2)$$

where $k = \frac{\omega}{c}$ is the acoustic wavenumber and c is the speed of sound.

Obviously, (1) describes a cylindrically symmetric sound field with the center being the SSD. Phase correct synthesis is restricted to sound fields whose local propagation direction coincides with that of the SSD. In practice, fixing the listening plane to $z = 0$ limits phase correct synthesis to 2D virtual sound fields invariant along the z-dimension and ensembles of 3D point sources located at the $z = 0$ plane. In the following sections the explicit requirement is formulated using the local wavenumber vector.

B. The Explicit solution: Spectral Division Method

The explicit solution exploits the fact that integral (1) describes a convolution along the x-axis. For the linear SSD geometry the orthogonal set of basis functions are given by exponentials and the spectral decomposition is obtained by performing a forward Fourier transform along the x-dimension. The spatial Fourier transform used in the present treatise is defined in the appendix A. In the spectral domain the convolution transforms into a multiplication and the wavenumber content of the synthesized field reads

$$\tilde{P}(k_x, y, 0, \omega) = \tilde{D}(k_x, \omega) \tilde{G}(k_x, y, 0, \omega). \qquad (3)$$

Here $\tilde{G}(k_x, y, 0, \omega)$ describes the spectrum of a 3D point source placed at the origin. Fixing the y-coordinate to a reference distance y_{ref} the driving function reads

$$\tilde{D}(k_x, \omega) = \frac{\tilde{P}(k_x, y_{\mathrm{ref}}, 0, \omega)}{\tilde{G}(k_x, y_{\mathrm{ref}}, 0, \omega)} \qquad (4)$$

and in the spatial domain it is given as

$$D(x_0, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\tilde{P}(k_x, y_{\mathrm{ref}}, 0, \omega)}{\tilde{G}(k_x, y_{\mathrm{ref}}, 0, \omega)} \mathrm{e}^{-\mathrm{j}k_x x_0} \mathrm{d}k_x. \qquad (5)$$

Since 3D point sources are applied for the synthesis in a 2D listening area the driving functions are termed as the 2.5 SDM driving functions. The driving functions ensure perfect synthesis of the pressure field on the reference line, however, the local propagation direction—and the particle velocity—can be reconstructed only for sound fields propagating in an in-plane direction in $z = 0$. Restricting the target sound field to fulfill these requirements reduces the base of the spectral decomposition to plane waves propagating parallel to the synthesis plane with $k_z = 0$. The dispersion relation for the 2.5D SDM scenario is formulated as $\left(\frac{\omega}{c}\right)^2 = k_x^2 + k_y^2$, therefore the k_x component completely determines the propagation of the actual plane wave component.

C. The Implicit solution: 2.5D WFS theory

Before the WFS solution is outlined two important concepts are introduced, utilized extensively in the followings.

1) Local Wavenumber Vector: Consider an arbitrary steady state harmonic sound field, written in a general polar form

$$P(\mathbf{x},\omega) = A(\mathbf{x},\omega) \mathrm{e}^{\mathrm{j}\phi(\mathbf{x},\omega)}, \qquad (6)$$

with $A(\mathbf{x},\omega), \phi(\mathbf{x},\omega) \in \mathbb{R}$ being its amplitude and phase functions. The dynamics of the wave propagation is described by the phase function of the sound field. Borrowed from geometrical optics we introduce the *local wavenumber vector*, defined by the gradient of the phase function [24]

$$\mathbf{k}_P^l(\mathbf{x}) = [k_x^l(\mathbf{x}),\ k_y^l(\mathbf{x}),\ k_z^l(\mathbf{x})]^{\mathrm{T}} = -\nabla\phi(\mathbf{x},\omega). \qquad (7)$$

The local wavenumber vector points in the direction of the maximum phase advance, i.e. it is perpendicular to the wave front in an arbitrary position. For an isotropic media this is the direction of the wave's energy flow, i.e. the wavenumber vector points in the local wave propagation direction. Utilizing the first order Taylor series of the phase function in (6) it can be shown, that each point of an arbitrary sound field can be approximated by local plane waves with the wavenumber given by the local wavenumber vector [24, Ch.7], for which the *local dispersion relation*

$$k^2 = \left(\frac{\omega}{c}\right)^2 = |\mathbf{k}_P^l(\mathbf{x})|^2 = k_x^l(\mathbf{x})^2 + k_y^l(\mathbf{x})^2 + k_z^l(\mathbf{x})^2 \qquad (8)$$

holds. The length of the vector is therefore frequency dependent, which remains unindicated in the followings for the sake of brevity.

For the sake of simplicity in the present article we restrict the investigation to strictly non-converging (i.e. to diverging/outgoing or plane waves), which can be defined using the local wavenumber vector as

$$\nabla \cdot \mathbf{k}_P^l(\mathbf{x}) = -\left(\phi''_{xx}(\mathbf{x},\omega) + \phi''_{yy}(\mathbf{x},\omega) + \phi''_{zz}(\mathbf{x},\omega)\right) \geq 0, \qquad (9)$$

where $\nabla\cdot$ is the divergence operator and $f''_{xx}(\mathbf{x})$ denotes the second partial derivative of a function f with respect

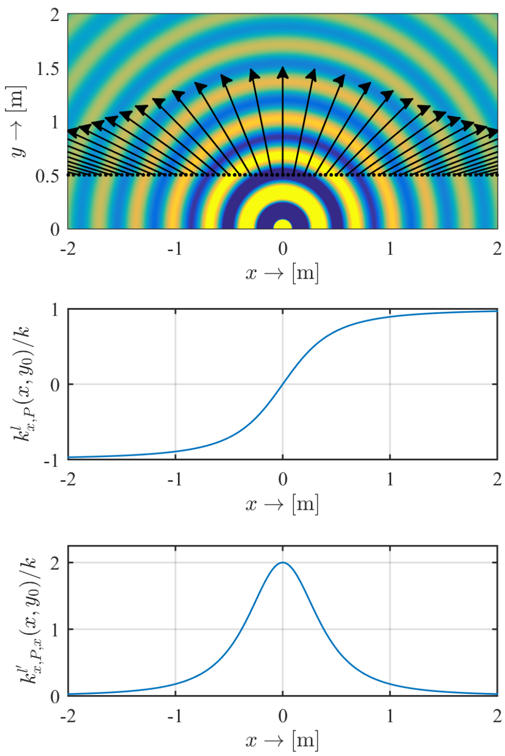

Fig. 2. Illustration of the wavenumber vector, in case of a 3D point source, taken along $y_0 = 0.5$ m. The wavenumber vector, and the x component of the local wavenumber along with its derivative (i.e. $\phi''_{xx}(x, y_0, 0, \omega)$) are normalized by $k = \frac{\omega}{c}$.

to x, taken at $\mathbf{x}$. As a more strict assumption, we require that inequality (9) is satisfied for each component separately, thus the second partial derivatives of the phase function are nonpositive. This holds trivially for simple sound fields, e.g. fields of point/line sources and plane waves. The possibility of synthesizing focused sources is therefore excluded from the present investigation. For an illustration of the introduced vector quantities see Figure 2.

2) Stationary Phase Approximation: Both 2.5D WFS theory and evaluation of Fourier integrals rely heavily on the *stationary phase approximation (SPA)*. As a basic tool of asymptotic analysis the method yields an approximate solution for integrals—containing at least one *critical point* in the integral path—of the form

$$I = \int_{-\infty}^{\infty} F(z)\,\mathrm{e}^{\mathrm{j}\phi(z)}\,\mathrm{d}z \tag{10}$$

when $\mathrm{e}^{\mathrm{j}\phi(z)}$ is highly oscillating and $F(z)$ is comparably slowly varying.

A rigorous derivation of the SPA based on integration by parts is given in [25]–[27]. More informally the method relies on the second order truncated Taylor series of the exponent around z^*, where $\phi'(z^*) = 0$ and $\phi''(z^*) \neq 0$, with $\phi'(z)$ denoting the derivative with respect to z:

$$\phi(z) \approx \phi(z^*) + \frac{1}{2}\phi''(z^*)(z - z^*)^2. \tag{11}$$

The critical point z^* is termed the *stationary point*. The SPA assumed that where the phase varies, i.e. $\phi'(z) \neq 0$, the integral of rapid oscillation cancels out, and the greatest contribution to the total integral comes from the immediate surroundings of the stationary point. Moreover in the proximity of the stationary point $F(z)$ can be regarded as constant with the value $F(z^*)$.

With these considerations—supposing also only one stationary point in the integration path—the integral becomes

$$I \approx F(z^*)\,\mathrm{e}^{+\mathrm{j}\phi(z^*)} \int_{-\infty}^{\infty} \mathrm{e}^{+\mathrm{j}\frac{1}{2}\phi''(z^*)(z-z^*)^2}\,\mathrm{d}z. \tag{12}$$

The remaining integral can be explicitly solved and the SPA of (10) becomes [25, (2.7.18)]

$$I \approx \sqrt{\frac{2\pi}{|\phi''(z^*)|}}\,F(z^*)\,\mathrm{e}^{+\mathrm{j}\phi(z^*)+\mathrm{j}\frac{\pi}{4}\,\mathrm{sgn}\left(\phi''(z^*)\right)}. \tag{13}$$

For multidimensional integrals a generalized SPA formulation is available with the stationary point found, where the phase gradient vanishes [25, (2.8.23)]. However, if the integration directions are orthogonal—as satisfied throughout the present treatise—the integration can be evaluated by applying multiple one dimensional SPAs consecutively.

3) 2.5D WFS driving functions: The basis of WFS theory is the *3D Neumann Rayleigh integral* formulation, representing an arbitrary sound field in the form of an infinite surface integral. Assume an infinite plane, located at $\mathbf{x}_0 = [x,\ 0,\ z]^{\mathrm{T}}$. Supposing, that all sources of sound are located at the $y < 0$ half space, at an arbitrary listening position with $y > 0$ the Rayleigh integral reads

$$P(\mathbf{x}, \omega) = -2 \iint_{-\infty}^{\infty} \left.\frac{\partial P(\mathbf{x}, \omega)}{\partial y}\right|_{\mathbf{x}=\mathbf{x}_0} G(\mathbf{x} - \mathbf{x}_0, \omega)\mathrm{d}z_0\,\mathrm{d}x_0. \tag{14}$$

This formulation implicitly contains the driving function for an infinite planar SSD.

In order to derive the driving functions for a linear SSD at $z = 0$ the SPA is applied to evaluate the 3D Rayleigh integral along the z-dimension. To find the phase function of the target field's y-derivative, consider the gradient of the general polar form (6)

$$\nabla P(\mathbf{x}, \omega) = \left(\frac{\nabla A_P(\mathbf{x}, \omega)}{A_P(\mathbf{x}, \omega)} + \mathrm{j}\nabla\phi_P(\mathbf{x}, \omega)\right) P(\mathbf{x}, \omega). \tag{15}$$

As a standard prerequisition a rapidly changing phase function was assumed compared to the amplitude rate of change, i.e. $\left|\frac{\nabla A_P(\mathbf{x},\omega)}{A_P(\mathbf{x},\omega)}\right| \ll |\nabla\phi_P(\mathbf{x}, \omega)| = k$ holds. In the high-frequency region therefore the gradient can be approximated by

$$\nabla P(\mathbf{x}, \omega) \approx -\mathrm{j}\mathbf{k}(\mathbf{x})P(\mathbf{x}, \omega). \tag{16}$$

Note, that this is again a local plane wave approximation of an arbitrary sound field.

Applying the HF gradient approximation in the Rayleigh integral (14) and exploiting, that the constant phase shift in (16) vanishes due to differentiation, the stationary phase position of (14) is found, where

$$\nabla_{x_0,z_0}\phi_P(\mathbf{x}_0, \omega) + \nabla_{x_0,z_0}\phi_G(\mathbf{x}_0 - \mathbf{x}, \omega) = 0, \tag{17}$$

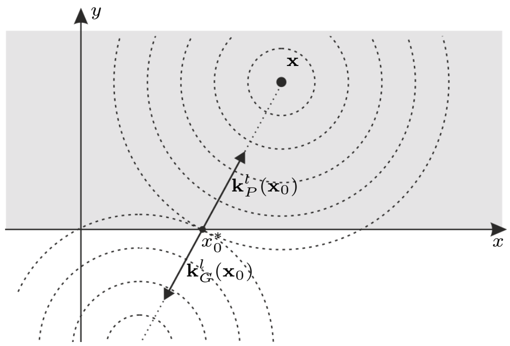

Fig. 3. Geometry for finding the stationary point for the 3D Rayleigh integral illustrated at the $z = 0$ plane.

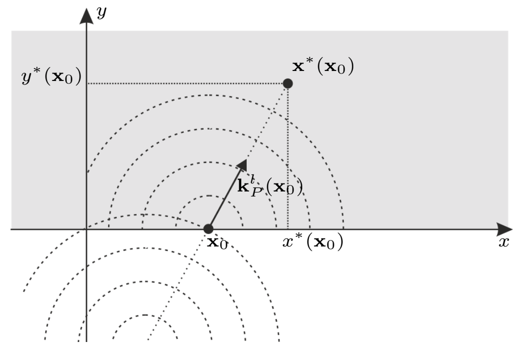

Fig. 4. Illustration for finding the points of correct synthesis for a given SSD element at $\mathbf{x}_0$, with a known virtual source local wavenumber vector

where the reciprocity of Green's function was exploited. Since the gradient of the fields represent the local wavenumber components $k^l_x(\mathbf{x}_0)$ and $k^l_z(\mathbf{x}_0)$, that along with the local dispersion relation completely determine the wavenumber vector the stationary position is found, where

$$\mathbf{k}^l_P(\mathbf{x}_0) = -\mathbf{k}^l_G(\mathbf{x} - \mathbf{x}_0) \tag{18}$$

i.e. where the local propagation direction of the virtual field and the Green's function coincide. For an illustration in the $z = 0$ plane refer to Figure 3. By replacing the Green's function into the SSD this means, that in the receiver position $\mathbf{x}$ the actual SSD element and the virtual field propagate into the same direction.

This interpretation is applied in order to approximate the 3D Rayleigh integral in the vertical dimension. Since it was assumed, that the virtual field propagates along the xy-axis at $z = 0$ (i.e. $k_{z,P}(x, y, 0) \equiv 0$), therefore fixing the receiver position to the same plane trivially yields the stationary point $z^* = 0$. Applying a one-dimensional SPA to (14) along the z-axis with accounting for the negative second phase derivatives yields the *2.5D Neumann Rayleigh integral*

$$P(\mathbf{x}, \omega) = -2 \int_{-\infty}^{\infty} \sqrt{\frac{2\pi}{\mathrm{j} \left| \phi''_{P,zz}(\mathbf{x}_0, \omega) + \phi''_{G,zz}(\mathbf{x} - \mathbf{x}_0, \omega) \right|}} \left. \frac{\partial P(\mathbf{x}, \omega)}{\partial y} \right|_{\mathbf{x}=\mathbf{x}_0} G(\mathbf{x} - \mathbf{x}_0, \omega) \mathrm{d}x_0, \tag{19}$$

with $\mathbf{x} = [x,\ y,\ 0]^{\mathrm{T}}$ and $\mathbf{x}_0 = [x_0,\ 0,\ 0]^{\mathrm{T}}$ denoting now in-plane positions. The 2.5 Rayleigh integral implicitly contains the 2.5D linear driving functions, yet depending on the receiver position.

In order to ensure optimal synthesis along an arbitrary *reference curve* instead of a point the role of receiver position and its stationary position is interchanged. According to the SPA each point on the SSD $\mathbf{x}_0$ dominates those parts of the listening plane $\mathbf{x}_{\mathrm{ref}}$, for which $\mathbf{x}_0$ is a stationary position, i.e. $k^l_{x,P}(\mathbf{x}_{\mathrm{ref}}(\mathbf{x}_0)) = k^l_{x,G}(\mathbf{x}_{\mathrm{ref}}(\mathbf{x}_0) - \mathbf{x}_0)$ is satisfied. Therefore, with choosing $\mathbf{x}_0$ as a free variable, within the validity of the SPA the 2.5D WFS driving functions can be extracted from (19) as

$$D_{2.5\mathrm{D\ WFS}}(\mathbf{x}_0, \omega) = -2 \times \sqrt{\frac{2\pi}{\mathrm{j} \left| \phi''_{P,zz}(\mathbf{x}_0, \omega) + \phi''_{G,zz}(\mathbf{x}_{\mathrm{ref}}(\mathbf{x}_0) - \mathbf{x}_0, \omega) \right|}} \left. \frac{\partial P(\mathbf{x}, \omega)}{\partial y} \right|_{\mathbf{x}=\mathbf{x}_0}. \tag{20}$$

In unified WFS theory the 2.5D correction factor was termed the *referencing function*

$$d_{\mathrm{ref}}(\mathbf{x}_0) = \frac{k}{\left| \phi''_{P,zz}(\mathbf{x}_0, \omega) + \phi''_{G,zz}(\mathbf{x}_{\mathrm{ref}}(\mathbf{x}_0) - \mathbf{x}_0, \omega) \right|}, \tag{21}$$

with normalization by k resulting in frequency independent referencing schemes [20]. Furthermore, it was shown, that the points of correct synthesis $\mathbf{x}_{\mathrm{ref}}(\mathbf{x}_0)$ are lying from each SSD elements towards the direction of the local wavenumber vector of the target field, thus

$$\mathbf{x}_{\mathrm{ref}}(\mathbf{x}_0) = \mathbf{x}_0 + \frac{\mathbf{k}^l_P(\mathbf{x}_0)}{k} d_0(\mathbf{x}_0), \tag{22}$$

where $d_0(\mathbf{x}_0)$ is the distance at which synthesis is optimized (for 2D virtual fields equaling with $d_{\mathrm{ref}}(\mathbf{x}_0)$). Substituting to the second derivative of the Green's function at $z = 0$ yields

$$\phi''_{G,zz}(\mathbf{x}_{\mathrm{ref}}(\mathbf{x}_0) - \mathbf{x}_0, \omega) = -\frac{k}{d_0(\mathbf{x}_0)}. \tag{23}$$

Thus, once the distance function $d_0(\mathbf{x}_0)$ is defined based on the actual shape of the desired reference curve $\mathbf{x}_{\mathrm{ref}}$ from (22), substitution into (23) and further into (20) yields the 2.5D WFS driving function with an arbitrary reference curve.

As a simple example: from geometrical considerations the reference distance for a fixed y_{ref} receiver coordinate—i.e. referencing the synthesis to a parallel *reference line*—is given by

$$d_0(\mathbf{x}_0) = y_{\mathrm{ref}} \frac{k}{k^l_{y,P}(\mathbf{x}_0)}, \tag{24}$$

and the general 2.5D WFS driving function ensuring optimal synthesis along a reference line reads

$$D_{2.5\mathrm{D\ WFS}}(\mathbf{x}_0, \omega) = -2 \sqrt{\frac{2\pi}{\mathrm{j} \left| \phi''_{P,zz}(\mathbf{x}_0, \omega) - \frac{k^l_{y,P}(\mathbf{x}_0)}{y_{\mathrm{ref}}} \right|}} \left. \frac{\partial P(\mathbf{x}, \omega)}{\partial y} \right|_{\mathbf{x}=\mathbf{x}_0}. \tag{25}$$

For a virtual 3D point source the formulation results in the well-known traditional 2.5D WFS driving functions [18].

III. Relation of the explicit and implicit solution

In this section the equivalence of the SDM and the unified WFS theory is presented. The main steps of the derivation are the followings

- First the general SDM driving functions are expressed in an asymptotic form. The calculus can be done by assuming, that the involved spectra are obtained via the SPA of the corresponding forward Fourier-transforms. This step links the spectral coefficients to stationary positions in the target field and the Green's function placed at the SSD.
- It is followed by the inverse Fourier transform of the asymptotic spectral driving functions. The evaluation of the inverse transform with the SPA relates the forward transform stationary positions to positions along the SSD. As a result purely spatial driving functions are obtained, calculated by the ratio of the target field and the Green's function taken in the corresponding stationary positions on a pre-defined receiver curve.
- Finally the latter driving functions are expressed in terms of the target field measured on the SSD. This is done by approximating the target field in the receiver position by the 3D Rayleigh integral, evaluated using the SPA.

Note, that the following derivation does not hold for a virtual plane wave, for which no unique stationary position can be found and also the SPA prerequisitions are not fulfilled for its spectrum. However, as a limiting case the final results hold for a virtual plane wave without any modification.

A. Asymptotic approximation of the SDM driving functions

The derivation starts from the general SDM driving functions given by (4). For the sake of brevity both frequency and z dependency is suppressed, since the driving functions are defined at $z = 0$. By definition the wavenumber content of the involved quantities is obtained via a forward Fourier transform

$$\tilde{P}(k_x, y) = \int_{-\infty}^{\infty} A_P(x,y) \mathrm{e}^{\mathrm{j}\phi_P(x,y)} \mathrm{e}^{\mathrm{j}k_x x} \mathrm{d}x, \quad (26)$$

$$\tilde{G}(k_x, y) = \int_{-\infty}^{\infty} A_G(x,y) \mathrm{e}^{\mathrm{j}\phi_G(x,y)} \mathrm{e}^{\mathrm{j}k_x x} \mathrm{d}x. \quad (27)$$

It is assumed, that the involved spectra are obtained by using the SPA: Under high-frequency assumptions the Fourier integral may be approximated by evaluation around its stationary point $x^*(k_x)$, where

$$-\phi'_x(x^*(k_x), y, \omega) = k^l_x(x^*(k_x), y, \omega) = k_x \quad (28)$$

holds: *the greatest contribution to the wavenumber spectrum at an arbitrary wavenumber k_x has the point in space x^*, where the local wavenumber component $k^l_x(x^*)$ equals the spectral coefficient k_x*. Note, that it is assumed, that in the sound field each local propagation direction is unique. The notation $x^*(k_x)$ indicates that each wavenumber component k_x determines a unique spatial stationary point.

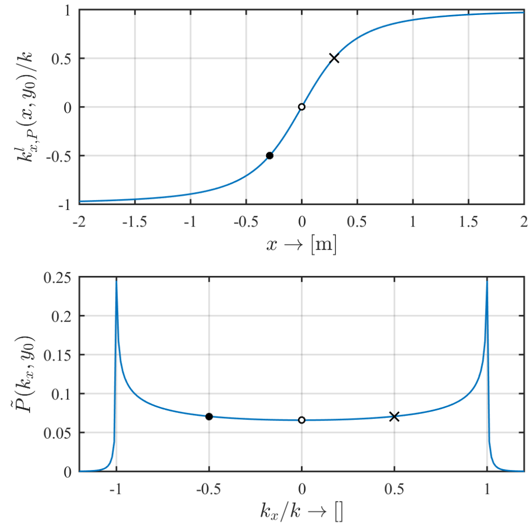

Fig. 5. Illustration of the relationship between the local wavenumber vector component $k^l_{x,P}(x, y_0)$ of the point source, shown in Figure 2 and its wavenumber content. The corresponding signs denote stationary point pairs in the spectral components and the local wavenumber components.

Supposing, that $x^*_P(k_x)$ and $x^*_G(k_x)$ are the stationary positions for the corresponding integrals, i.e.

$$k^l_{x,P}(x^*_P(k_x), y, \omega) = k^l_{x,G}(x^*_G(k_x), y, \omega) = k_x \quad (29)$$

holds, and accounting for the negative second derivatives—since both P and G are non-converging waves—their spectra can be approximated as [28, Ch. 5]

$$\tilde{P}(k_x, y) \approx \sqrt{\frac{2\pi}{\mathrm{j}|\phi''_{P,xx}(x^*_P(k_x), y)|}} A_P(x^*_P(k_x), y) \mathrm{e}^{\mathrm{j}\phi_P(x^*_P(k_x),y)} \mathrm{e}^{\mathrm{j}k_x x^*_P(k_x)}, \quad (30)$$

$$\tilde{G}(k_x, y) \approx \sqrt{\frac{2\pi}{\mathrm{j}|\phi''_{G,xx}(x^*_G(k_x), y)|}} A_G(x^*_G(k_x), y) \mathrm{e}^{\mathrm{j}\phi_G(x^*_G(k_x),y)} \mathrm{e}^{\mathrm{j}k_x x^*_G(k_x)}. \quad (31)$$

The asymptotic approximation of the SDM driving functions on a given spectral component therefore reads

$$\tilde{D}(k_x) \approx \sqrt{\frac{|\phi''_{G,xx}(x^*_G(k_x), y)|}{|\phi''_{P,xx}(x^*_P(k_x), y)|}} \frac{P(x^*_P(k_x), y)}{G(x^*_G(k_x), y)} \mathrm{e}^{\mathrm{j}k_x(x^*_P(k_x) - x^*_G(k_x))}. \quad (32)$$

B. SDM driving functions in the spatial domain

Using the asymptotic approximation of the SDM spectrum the spatial driving functions are obtained via an inverse spatial Fourier transform:

$$D(x_0) \approx \frac{1}{2\pi} \int_{-\infty}^{\infty} \sqrt{\frac{|\phi''_{G,xx}(x^*_G(k_x), y)|}{|\phi''_{P,xx}(x^*_P(k_x), y)|}} \frac{P(x^*_P(k_x), y)}{G(x^*_G(k_x), y)} \mathrm{e}^{\mathrm{j}k_x(x^*_P(k_x) - x^*_G(k_x))} \mathrm{e}^{-\mathrm{j}k_x x_0} \mathrm{d}k_x. \quad (33)$$

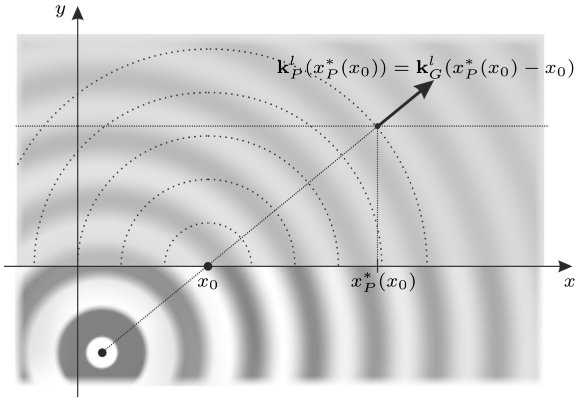

Fig. 6. Illustration of the evaluation position $x_P^*(x_0)$ (and $x_G^*(x_0)$) as the function of x_0. For a given SSD position x_0 the stationary positions at y, where the virtual field propagation direction coincides with that of the Green's function translated into x_0. At $x_P^*(x_0)$ the local curvature of the translated Green's function is always greater, than that of the virtual field.

Again, the integral is approximated using the SPA, with the phase function under investigation given by

$$\tilde{\phi}(k_x) = \phi_P(x_P^*(k_x), y) - \phi_G(x_G^*(k_x), y) \\ + k_x\left(x_P^*(k_x) - x_G^*(k_x)\right) - k_x x_0. \quad (34)$$

Similarly to the forward transform, in the inverse transform each wavenumber component k_x will dominate one spatial position x_0, where the actual wavenumber component coincides with the local wavenumber of the sound field $k_x^l(x_0)$. This wavenumber is found as the stationary phase wavenumber of integral (33) [28]. Differentiating (34) with respect to k_x and exploiting (28) yields the stationary wavenumber $k_x^*(x_0)$, satisfying

$$\left.\frac{\partial \tilde{\phi}(k_x)}{\partial k_x}\right|_{k_x = k_x^*(x_0)} = x_P^*(k_x^*(x_0)) - x_G^*(k_x^*(x_0)) - x_0 = 0. \quad (35)$$

Note, that this definition relates the evaluation points x_P^* and x_G^* directly to the actual SSD coordinate x_0, therefore the intermediate stationary wavenumber ($k_x^*()$) dependency may be omitted. Furthermore along with the definition of the forward transform stationary points (29), the stationary positions satisfy

$$k_{x,P}^l(x_P^*(x_0), y, \omega) = k_{x,G}^l(x_G^*(x_0), y, \omega), \quad (36)$$

where

$$x_G^*(x_0) = x_P^*(x_0) - x_0 \quad (37)$$

holds. This results states, that *for a given SSD coordinate x_0 the evaluation point x_P^* is found, where the local propagation direction of the target field P coincides with that of a point source positioned at $[x_0,\ 0,\ 0]^{\mathrm{T}}$ measured on the reference line*. For an illustration refer to Figure 6.

In order to evaluate the SPA of the inverse integral (33) one still needs the second derivative of the phase function (34) with respect to k_x. Performing the differentiation and using (29) and its derivative to express the required rate of change of the stationary positions ($x_P^{*'}(k_x)$ and $x_G^{*'}(k_x)$) the second derivative reads

$$\frac{\partial^2}{\partial k_x^2}\tilde{\phi}(k_x) = \frac{\phi''_{P,xx}(x_P^*(k_x), y) - \phi''_{G,xx}(x_G^*(k_x), y)}{\phi''_{P,xx}(x_P^*(k_x), y)\phi''_{G,xx}(x_G^*(k_x), y)}. \quad (38)$$

The absolute value of the phase function's second derivatives are proportional to the curvature of the wavefront in an arbitrary position [29], which is maximal for a point source. Furthermore, from simple geometrical considerations the curvature—since being inversely proportional with the distance from the point source— is maximal for the actual SSD element at x_0 in the stationary receiver position at $x_P^*(x_0)$ (refer to Figure 6 for an illustration). Accounting also for the negative sign for simple virtual fields

$$\phi''_{P,xx}(x_P^*(k_x), y) > \phi''_{G,xx}(x_G^*(k_x), y) \quad (39)$$

holds and the sign of (38) is positive.

These results now may be substituted back into the SPA (13) of the inverse transform (33). For the sake of brevity in the followings the evaluation point is denoted by $x_P^* \rightarrow x^*$. Based on (35) around the stationary wavenumber the exponentials cancel out, and accounting for the positive sign of the second derivative as a result one obtains the spatial asymptotic SDM driving functions

$$D(x_0) \approx \sqrt{\frac{\left|\phi''_{G,xx}(x^*(x_0) - x_0, y)\right|^2}{\left|\phi''_{P,xx}(x^*(x_0), y) - \phi''_{G,xx}(x^*(x_0) - x_0, y)\right|}} \\ \sqrt{\frac{\mathrm{j}}{2\pi}}\frac{P(x^*(x_0), y)}{G(x^*(x_0) - x_0, y)}, \quad (40)$$

where $k_{x,P}^l(x^*(x_0)) = k_{x,G}^l(x^*(x_0) - x_0)$ holds. This result states, that an arbitrary sound field may be synthesized by finding the positions along the reference line, where the propagation direction/wavefront of the target field matches the field of the actual SSD elements. In this stationary position the driving functions are obtained by the ratio of the target field and the actual SSD element, corrected by the factor, containing the wavefront curvature at the same position. Therefore the explicit, global solution can be approximated by simple local wavefront matching.

One important fact is pointed out here: although having derived the above driving functions in terms a forward an inverse spatial Fourier transform along a straight line, there is no restriction on the y-coordinate of the stationary point in (40) due to the local approximations involved: the y-coordinate might be x_0-dependent. This means, that an arbitrary referencing curve may be defined as $\mathbf{x}^*(x_0)$, and the driving functions can be calculated by finding the stationary positions $k_{x,P}^l(\mathbf{x}^*(x_0)) = k_{x,G}^l(\mathbf{x}^*(x_0) - \mathbf{x}_0)$ along this curve. Evaluating the driving functions in the stationary positions will result in amplitude correct synthesis along the reference curve. This means, that the presented driving functions are equivalent to the 2.5D WFS driving functions, written in terms of the target pressure field taken on the reference curve. In the followings first a simple example is presented in order to demonstrate the validity of the spatial SDM driving functions.

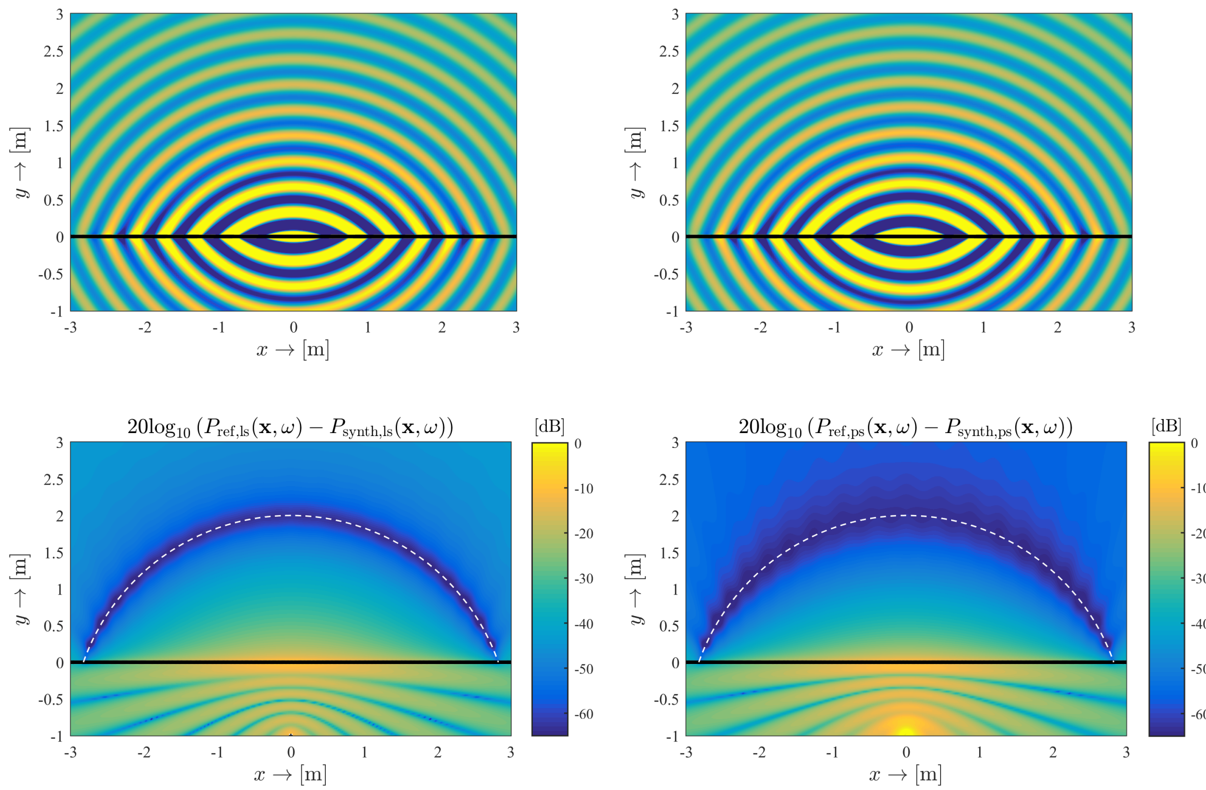

Fig. 7. Synthesis of a 2D point source (a,c) and a 3D point source (b,d) located at $\mathbf{x}_{\mathrm{s}} = [0,\ -1,\ 0]^{\mathrm{T}}$, oscillating at $\omega_0 = 2\pi \cdot 1$ krad/s. The synthesis is referenced on a circle around the virtual source, with a radius of $R_{\mathrm{ref}} = 3$ m.

C. Spatial SDM application example

Consider the synthesis of a 2D/3D point source, referencing the synthesis to a circle around the virtual point source. For the sake of simplicity the source is located at $\mathbf{x}_{\mathrm{s}} = [0,\ y_s,\ 0]^{\mathrm{T}}$. Along with the equation describing the reference curve $\mathbf{x}^*(x_0) = [x^*(x_0),\ y^*(x_0),\ 0]^{\mathrm{T}}$ the stationary points satisfy the following equations (since under high-frequency approximations the phase function of a 2D and 3D point source equal [27])

$$\frac{x^*(x_0)}{\sqrt{x^*(x_0)^2 + (y^*(x_0) - y_s)^2}} = \frac{x^*(x_0) - x_0}{\sqrt{(x^*(x_0) - x_0)^2 + y^*(x_0)^2}}, \tag{41}$$

$$x^*(x_0)^2 + (y^*(x_0) - y_s)^2 = R_{\mathrm{ref}}^2. \tag{42}$$

The solution for the equations is given by

$$x^*(x_0) = x_0 \frac{R_{\mathrm{ref}}}{\sqrt{x_0^2 + y_s^2}} \tag{43}$$

$$y^*(x_0) = y_s \left(1 - \frac{R_{\mathrm{ref}}}{\sqrt{x_0^2 + y_s^2}}\right) \tag{44}$$

Substituting into (40) yields the explicit driving function in the spatial domain referencing the synthesis on a circle. In this case the only difference between the 2D and 3D point source driving functions is the actual form of $P(x^*(x_0), y^*(x_0))$.

Investigating Figure 7 verifies, that in both cases the synthesis is optimized on the prescribed reference curve.

D. The Rayleigh SDM formulation

As a last step in order to present the relation between SDM and WFS the spatial domain SDM driving function (40) is expressed in terms of the target field, measured on the SSD. This is done by expressing the target field at the evaluation point in terms of a 3D Neumann Rayleigh integral, evaluated by the SPA. For the derivation of the 2.5D Rayleigh integral (19) the vertical SPA had already been evaluated. In order to evaluate the the horizontal integral the stationary position $\mathbf{x}_0^*(\mathbf{x})$ is found, where

$$k_{x,P}^l(\mathbf{x}_0^*(\mathbf{x})) = k_{x,G}^l(\mathbf{x} - \mathbf{x}_0^*(\mathbf{x})) \tag{45}$$

holds, with $\mathbf{x} = [x,\ y,\ 0]^{\mathrm{T}}$ and $\mathbf{x}_0^*(\mathbf{x}) = [x_0^*(x),\ 0,\ 0]^{\mathrm{T}}$. Around the stationary point the the asymptotic approximation of the 3D Rayleigh integral reads

$$P(\mathbf{x}) \approx -2\frac{2\pi}{\mathrm{j}} \frac{1}{\sqrt{\left|\phi''_{P,zz}(\mathbf{x}_0^*(\mathbf{x})) + \phi''_{G,zz}(\mathbf{x} - \mathbf{x}_0^*(\mathbf{x}))\right|}} \times \frac{1}{\sqrt{\left|\phi''_{P,xx}(\mathbf{x}_0^*(\mathbf{x})) + \phi''_{G,xx}(\mathbf{x} - \mathbf{x}_0^*(\mathbf{x}))\right|}} \times \left.\frac{\partial P(\mathbf{x})}{\partial y}\right|_{\mathbf{x}=\mathbf{x}_0^*(\mathbf{x})} G(\mathbf{x} - \mathbf{x}_0^*(\mathbf{x})), \tag{46}$$

One still needs to express the second derivative of the target field phase ϕ_P at an arbitrary receiver position $\mathbf{x}$ in terms of the second derivative taken on the SSD at the stationary point

$\mathbf{x}_0^*(\mathbf{x})$. This is possible by expressing the second derivative of the asymptotic Rayleigh integral's phase function, i.e. from

$$\frac{\partial^2}{\partial x^2}\left(\phi_P(\mathbf{x}_0^*(\mathbf{x})) + \phi_G(\mathbf{x} - \mathbf{x}_0^*(\mathbf{x}))\right). \quad (47)$$

Again, performing the differentiation by applying the chain rule, exploiting the Rayleigh stationary point definition (45) for simplification and its derivative to express $\mathbf{x}_{0,x}^{*'}(\mathbf{x})$ yields

$$\phi''_{P,xx}(\mathbf{x}) \approx \frac{\phi''_{P,xx}(\mathbf{x}_0^*(\mathbf{x}))\phi''_{G,xx}(\mathbf{x} - \mathbf{x}_0^*(\mathbf{x}))}{\phi''_{P,xx}(\mathbf{x}_0^*(\mathbf{x})) + \phi''_{G,xx}(\mathbf{x} - \mathbf{x}_0^*(\mathbf{x}))}. \quad (48)$$

Comparing the definition of the stationary points for the Rayleigh integral (45) and the stationary SDM evaluation points (36) is is revealed, that they describe stationary point pairs. Therefore in the spatial SDM driving function (40) both the target field and the second phase derivative at the evaluation point $x^*(x_0)$ can be expressed by the corresponding quantities taken at x_0, by using (46) and (48) respectively with a substitution of $\mathbf{x}_0^*(\mathbf{x}) \rightarrow [x_0,\ 0,\ 0]^{\mathrm{T}}$ and $\mathbf{x} \rightarrow [x^*(x_0),\ y,\ 0]^{\mathrm{T}}$.

By substituting (48) into (40) yields

$$D(x_0) \approx \sqrt{\left|\phi''_{P,xx}(x_0, 0) + \phi''_{G,xx}(x^*(x_0) - x_0, y)\right|} \sqrt{\frac{\mathrm{j}}{2\pi}} \frac{P(x^*(x_0), y)}{G(x^*(x_0) - x_0, y)} \quad (49)$$

and finally by expressing the target field by (46) the Green's function vanishes and one obtains the asymptotic form of the spatial SDM driving functions written in terms of the target field taken on the SSD

$$D(x_0) \approx -2\sqrt{\frac{2\pi}{\mathrm{j}}} \frac{1}{\sqrt{\left|\phi''_{P,zz}(x_0, 0) + \phi''_{G,zz}(x^*(x_0) - x_0, y)\right|}} \left.\frac{\partial P(\mathbf{x})}{\partial y}\right|_{\mathbf{x}=\mathbf{x}_0}. \quad (50)$$

Comparing with (20) reveals, that the asymptotic SDM driving functions exactly coincide the 2.5D WFS driving function. It is important to note however, that the WFS driving functions were obtained from the 2.5D Neumann Rayleigh integral in an intuitive manner, by introducing the reference curve concept with interchanging the role of the receiver position and its stationary SSD position. On the other hand the driving function (50) inherently contains the horizontal SPA and the reference curve concept. It is therefore verified, that under high-frequency assumptions, WFS is the asymptotic or local approximation of the global explicit solution.

IV. Conclusion

The article presented a detailed treatise on the connection of the well-established implicit SFS technique Wave Field Synthesis and the explicit inverse solution termed the Spectral Division Method. Emerging from the physical interpretation of the involved approximations it was shown explicitly, that Wave Field Synthesis constitutes a local wavefront matching solution for the underlying problem by applying local plane wave approximations for the wave field to be synthesized. Applying WFS the virtual and synthesized wavefronts are matched both in propagation direction and wavefront curvature. For the sake of better comparability a (so far unknown) generalization for the 2.5D WFS driving functions was presented, being able to synthesize an arbitrary 3D virtual field, optimized on an arbitrary curve, termed as the reference curve.

SDM—constituting a global mode-matching solution—provides an explicit solution, ensuring perfect synthesis along a reference line, although yielding the driving function in terms of a multiple integral, making practical applications unfeasible. It was proven, that assuming high-frequency conditions, within the validity of the local plane wave approximation of the target sound field, the SDM can be expressed in the spatial domain, providing a wavefront matching solution for the SFS problem in terms of the target field and the Green's function taken at the desired reference curve. The presented asymptotic driving functions—so far unknown for the sound field reproduction problem—gives the possibility to control an arbitrary sound field on a reference curve, without having information on the sound field along the SSD. This option may be feasible for e.g. suppressing the diffraction effects due to truncated linear SSDs. The validity of the presented new driving functions are also verified via simple simulation examples.

Finally the general high-frequency/local equivalence of WFS and the SDM is shown by approximating the target field on the reference field by the asymptotic approximation of the Rayleigh integral.

Appendix A
Definition of Fourier transforms

Throughout the article the temporal and spatial forward Fourier transforms are defined conventionally as

$$P(\mathbf{x}, \omega) = \int_{-\infty}^{\infty} p(\mathbf{x}, t)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t, \quad (51)$$

$$\hat{P}(k_x, y, z, \omega) = \int_{-\infty}^{\infty} P(x, y, z, \omega)\mathrm{e}^{\mathrm{j}k_x x}\mathrm{d}x. \quad (52)$$

The corresponding inverse transforms are defined with reversing the exponential sign and normalizing by $\frac{1}{2\pi}$.

References

[1] S. Spors, H. Wierstorf, A. Raake, F. Melchior, M. Frank, and F. Zotter, "Spatial sound with loudspeakers and its perception: A review of the current state," *Proceedings of the IEEE*, vol. 101, no. 9, pp. 1920–1938, 2013.

[2] J. Ahrens, "The single-layer potentital approach applied to sound field synthesis including of non-enclosing distributions of secondary sources," Ph.D. dissertation, Technische Universität Berlin, 2010.

[3] F. M. Fazi, "Sound field reproduction," Ph.D. dissertation, University of Southampton, 2010.

[4] J. Ahrens and S. Spors, "An analytical approach to 2.5d sound field reproduction employing circular distributions of non-omnidirectional loudspeakers," in *IEEE 17th European Signal Processing Conference*, Glasgow, August 2009.

[5] ——, "An analytical approach to sound field reproduction using circular and spherical loudspeaker distributions," *Acta Acus. Utd. With Acust.*, vol. 94, pp. 988–999, 2008.

[6] M. A. Poletti, "Three-dimensional surround sound systems based on spherical harmonics," *J. Audio Eng. Soc.*, vol. 53, no. 11, pp. 1004–1025, 2005.

[7] F. Zotter, “Analysis and synthesis of sound-radiation with spherical arrays,” Ph.D. dissertation, Institute of Electronic Music and Acoustics University of Music and Performing Arts, 2009.
[8] M. A. Gerzon, “Periphony: With-height sound reproduction,” *J. Acoust. Soc. Am.*, vol. 21, pp. 2–10, Jan./Feb. 1973.
[9] J. Ahrens and S. Spors, “Reproduction of a plane-wave sound field using planar and linear arrays of loudspeakers,” in *Proc. of 3rd Intl. Symposium on Communications, Control and Signal Processing (IEEE 3rd ISCCSP), Malta*, 2008, pp. 1486–1491.
[10] ——, “Sound Field Reproduction Using Planar and Linear Arrays of Loudspeakers,” *IEEE Trans. Audio, Speech, Lang. Process.*, vol. 18, no. 8, pp. 2038–2050, Nov. 2010.
[11] ——, “Applying the ambisonics approach to planar and linear distributions of secondary sources and combinations thereof,” *Acta Acus. Utd. With Acust.*, vol. 98, pp. 28–36, 2012.
[12] J. Ahrens, *Analytic Methods of Sound Field Synthesis*, 1st ed. Berlin: Springer, 2012.
[13] D. de Vries, *Wave Field Synthesis – AES Monograph*. New York: Audio Engineering Society, 2009.
[14] A. Berkhout, “A holographic approach to acoustic control,” *J. Audio Eng. Soc.*, vol. 36, no. 12, pp. 977–995, December 1988.
[15] A. Berkhout, D. de Vries, and P. Vogel, “Acoustic control by wave field synthesis,” *J. Acoust. Soc. Am.*, vol. 93, no. 5, pp. 2764–2778, 1993.
[16] P. Vogel, “Application of wave field synthesis in room acoustics,” Ph.D. dissertation, Delft University of Technology, 1993.
[17] E. Start, “Direct sound enhancement by wave field synthesis,” Ph.D. dissertation, Delft University of Technology, 1997.
[18] E. Verheijen, “Sound reproduction by wave field synthesis,” Ph.D. dissertation, Delft University of Technology, 1997.
[19] S. Spors, R. Rabenstein, and J. Ahrens, “The Theory of Wave Field Synthesis Revisited,” in *Proc. of the 124th Audio Eng. Soc. Convention*, Amsterdam, May 2008.
[20] G. Firtha, P. Fiala, F. Schultz., and S. Spors, “Referencing schemes for wave field synthesis driving functions: A unified theory,” *IEEE Trans. Audio, Speech, Lang. Process.*, 2016, submitted for peer-review.
[21] S. Spors and J. Ahrens, “Analysis and Improvement of Pre-Equalization in 2.5-dimensional Wave Field Synthesis,” in *Proc. of the 128th Audio Eng. Soc. Convention*, London, May 2010.
[22] F. Schultz, “Sound Field Synthesis for Line Source Array Applications in Large-Scale Sound Reinforcement,” Ph.D. dissertation, University of Rostock, 2016.
[23] F. Schultz and S. Spors, “On the Connections of Wave Field Synthesis and Spectral Division Method Plane Wave Driving Functions,” in *Fortschritte der Akustik: Tagungsband d. 42. DAGA, Aachen*, 2016.
[24] H. Römer, *Theoretical Optics: An Introduction*. Wiley-VCH, 2005.
[25] N. Bleistein, *Mathematical Methods for Wave Phenomena*. Academic Press, 1984.
[26] N. Bleistein and R. Handelsman, *Asymptotic Expansions of Integrals*. Dover Publications, 1986.
[27] E. G. Williams, *Fourier Acoustics: Sound Radiation and Nearfield Acoustical Holography*, 1st ed. London: Academic Press, 1999.
[28] E. Tracy, A. Brizard, A. Richardson, and A. Kaufman, *Ray tracing and beyond: Phase Space Methods in Plasma Wave Theory*. Cambridge University Press, 2014.
[29] H. P. Herzig and R. Dandliker, “Holographic optical scanning elements: analytical method for determining the phase function,” *J. Opt. Soc. Am. A*, vol. 4, no. 6, pp. 1063–1070, June 1987.