Prefiltering the wave field synthesis operators - anti-aliasing and source directivity

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Abstract

Wave field synthesis and spectral division method are two well-known sound field reproduction techniques. Their aim is to reproduce an arbitrary sound field in an extended listening area by driving densely spaced loudspeakers (secondary sources) with a properly derived driving function. In the present paper the physical interpretation of linear prefiltering of the driving function is investigated. It is revealed here, that filtering the driving functions can be interpreted as modifying the spatial and spectral properties of the virtual and secondary sources. To utilize the presented approach two applications are discussed. It is shown here, that applying linear filtering arbitrary virtual source directivity can be realized directly. In case of spatially sampled secondary source distribution proper low-pass filter design is needed to avoid spatial aliasing. To accomplish that two possible filter design methods are presented, applying raised cosine and Chebyshev-filters.

1 Introduction

Spatial sound reproduction systems with a large number of densely spaced loudspeakers have been used increasingly in the last decade. The aim of these systems is to physically reproduce the sound field of a virtual sound source in an extended listening area. The techniques to achieve this are termed as sound field reproduction techniques. Wave field synthesis (WFS), spectral division method (SDM) and nearfield compensated higher order ambisonics (HOA) are the most commonly known analytical methods, while there exist numerical solutions as well, like multichannel inversion.

Traditional WFS and SDM assumes that the sources used for reproduction form a continuous source distribution along an infinite line that bounds the reproduction area. The amplitude and phase function of source elements is termed as the driving function. Wave field synthesis is based on the Rayleigh integrals and is used to derive the driving functions in the spatial domain directly, while spectral division method operates in the Fourier-domain, so for a directly implementable result an additional inverse transform is needed. Both techniques result in a driving function, that can be regarded as a one-dimensional complex valued continuous function along a spatial axis.

Previous studies on the topic (e.g [1, 2]) dealt with the comparison of driving functions derived by WFS and SDM for the case of a virtual plane wave and two-dimensional synthesis. In section 2.2 the two methodologies are compared for the case of a virtual monopole source in three-dimensions.

Linear filtering of the driving functions can be defined both in spatial and wave number domains. To the knowledge of the authors the effects of filtering have not been investigated in details up to now. As the most general case we will examine, how the linear filtering of the driving function, derived for a virtual monopole source can be interpreted physically.

In the traditional derivation of WFS driving functions, the possibility of synthesizing directive sources was given in [3]. However, the applied approximations restricted the application to the far-field of the virtual source. In the present paper we give a treatise on synthesizing directive sources by linear filtering of the monopole driving function. We will see that both synthesizing an extended source and synthesizing a multipole can be realized by applying a properly chosen linear filter to the monopole driving function.

As a further application the effects of spatial sampling are investigated. Without proper prefiltering, the field synthesized by a sampled source distribution will contain aliasing components. Applying the anti-aliasing approaches given in the related literature (e.g in [4, 5]) results in serious artifacts in the near field of the virtual source. In the last section of the paper we will discuss, what considerations have to be made in order to properly design the anti-aliasing filter.

2 Theory of sound field reproduction

This section first introduces the basic theory of sound field reproduction methods, namely the wave field synthesis (WFS) and spectral division method (SDM), followed by the theory and physical interpretation of linearly filtered synthesis functions. As the author's contribution, the general equivalence of WFS and SDM is shown. The formulas shown here hold for linear source distributions only.

2.1 Derivation of synthesis operators

2.1.1 The traditional WFS driving functions

The mathematical basis of traditional WFS is the Rayleigh I integral, a special case of Kirchhoff-Helmholtz integral formula [6, 7, 8]. The Rayleigh integral describes the wave field of an infinite plane radiating to the infinite halfspace, so Sommerfeld radiation condition is fulfilled.

In this special configuration the generated wave field can be written as the sum of the wave fields of individual monopoles, called *secondary sources*, that form the *secondary source distribution*. In the aspect of WFS, the Rayleigh I integral states that the pressure field of a *virtual source* behind an infinite plane can be synthesized in front of the plane with a planar monopole distribution, driven with two times the normal velocity component created by the virtual source on the plane. Denoting $\mathbf{x}_0 = [x\ 0\ z]^{\mathrm{T}}$ the position vector of secondary sources the integral reads

$$P(\mathbf{x}, \omega) = \int_S \underbrace{2\,\frac{\partial}{\partial \mathbf{n}} P(\mathbf{x}_0, \omega)}_{Q_{3\mathrm{D}}(\mathbf{x}_0,\omega)}\, G_{3\mathrm{D}}(\mathbf{x}|\mathbf{x}_0, \omega) \mathrm{d}S, \tag{1}$$

where $Q_{3\mathrm{D}}(\mathbf{x}_0, \omega) = \mathrm{j}\omega V_{\mathrm{n}}(\mathbf{x}_0, \omega)$ is the three-dimensional *synthesis operator*, or *driving function*, while $G(\mathbf{x}|\mathbf{x}_0, \omega)$ is the free-field Green's function of the linear wave equation [9, 10]. In three-dimensions the Green's function is given by

$$G_{3\mathrm{D}}(\mathbf{x}|\mathbf{x}_0, \omega) = \frac{1}{4\pi} \frac{\mathrm{e}^{-\mathrm{j}k|\mathbf{x}-\mathbf{x}_0|}}{|\mathbf{x} - \mathbf{x}_0|}, \tag{2}$$

which is the sound field generated by a point source located in $\mathbf{x}_0$.

Practical implementations of WFS employ loudspeakers located along a horizontal line instead of a plane. The field of loudspeakers can be modelled as the field of a continuous linear distribution of three-dimensional monopoles along an infinite line, called the *synthesis line*. The geometry is depicted in figure 1. The effect of limiting the secondary sources to the straight line is that sound synthesis can only be performed in the horizontal plane containing the source distribution, called *synthesis plane*. This means that we deal with two-dimensional synthesis in three-dimensional space, employing three-dimensional secondary sources. This

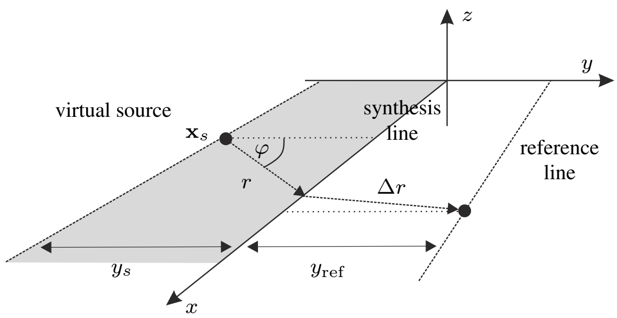

Figure 1: Geometry for the derivation of 2.5-dimensional synthesis operator in spatial and wave number domain

type of synthesis is referred as 2.5-dimensional synthesis. The synthesized sound field in this case can be written as a line integral:

$$P(\mathbf{x},\omega) = \int_{-\infty}^{\infty} Q_{2.5\mathrm{D}}(\mathbf{x}_0,\omega)\frac{\mathrm{e}^{-\mathrm{j}k|\mathbf{x}-\mathbf{x}_0|}}{4\pi|\mathbf{x}-\mathbf{x}_0|}\mathrm{d}x. \tag{3}$$

The driving function $Q_{2.5\mathrm{D}}(x,\omega)$ may be derived by employing the the so-called stationary phase approximation [11, 12], as given by Start and Verheijen in [13, 3], or by applying the the large-argument asymptotic expansion of Hankel functions [14, 15], as presented by Spors and Ahrens in [4, 16]. Both derivations apply far-field approximations ($kr \gg 1$), therefore the synthesis operators can only reconstruct the desired wave field far from the synthesis line or in the high-frequency region phase correctly. Due to the approximations amplitude errors occur even in the high-frequency region, the driving functions can only be optimized to a line, parallel to the secondary source distribution, called *reference line*, denoted by $\mathbf{x}_{\mathrm{ref}} = [x\ y_{\mathrm{ref}}\ 0]^{\mathrm{T}}$. On this line amplitude and phase correct synthesis is possible.

According to [3] the driving function for a virtual monopole at $\mathbf{x}_{\mathrm{s}} = [x_s\ y_s\ 0]^{\mathrm{T}}$ with the notation seen in figure 1 reads

$$Q_{2.5\mathrm{D}}(\mathbf{x}_0,\omega) = \sqrt{\frac{\mathrm{j}k}{2\pi}}\sqrt{\frac{|y_{\mathrm{ref}}|}{|y_{\mathrm{ref}}|+|y_{\mathrm{s}}|}}\cos\varphi\frac{\mathrm{e}^{-\mathrm{j}kr}}{\sqrt{r}}. \tag{4}$$

This formulation – which will be used throughout this paper – is more significant than the one given by Spors [16, 1] as it ensures amplitude correct synthesis along a straight line, while the Spors type formulation is not referenced to a line but to a circle around the individual secondary sources [1]. As the driving function is defined only on line $[x\ 0\ 0]^{\mathrm{T}}$ and fixed frequency is assumed, the argument of the driving function will be denoted as $Q_{2.5\mathrm{D}}(x)$ hereafter.

2.1.2 Driving functions in the wave number domain

For a continuous pressure distribution along an infinite line as a function of x we can define its one-dimensional spatial Fourier transform:

$$\tilde{P}(k_x,y,z,\omega) = \int_{-\infty}^{\infty} P(\mathbf{x},\omega)\mathrm{e}^{\mathrm{j}k_x x}\mathrm{d}x \tag{5}$$

and the inverse Fourier transform reads as

$$P(\mathbf{x},\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{P}(k_x,y,z,\omega)\mathrm{e}^{-\mathrm{j}k_x x}\mathrm{d}k_x. \tag{6}$$

Thus the Fourier transform pair of x is $k_x = k\sin\varphi$, the x component of the wave number $k = \omega/c$. The physical interpretation of the spectrum is that each value of $\tilde{P}(k_x,y,z,\omega)$ represents a monochromatic plane wave with the wave number k, arriving to the examined line under the angle of inclination φ. As the WFS operators are defined as the normal velocity excited by the virtual source along the synthesis line, the spectrum of a plane wave synthesis operator is a Dirac delta function. For use in 2.5-dimensional synthesis, we will restrict the investigation to the synthesis plane ($z = 0$), therefore position along z-axis will not be denoted in the followings. Assuming that the examination is performed on a given frequency, denotion of ω dependency will be also omitted.

Let's notice that integral (3) represents a one-dimensional convolution along the synthesis line $\mathbf{x}_0 = [x_0\, 0\, 0]^{\mathrm{T}}$, that can be written on the reference line y_{ref} as

$$P(x, y_{\mathrm{ref}}) = \int_{-\infty}^{\infty} Q(x_0) G(x - x_0, y_{\mathrm{ref}}) \mathrm{d}x_0 = Q(x) \otimes G(x, y_{\mathrm{ref}}). \tag{7}$$

The kernel of convolution is the shift invariant Green's function that gives the impulse response of each point on the synthesis line with respect to the reference line. According to the convolution theorem, convolution is transformed into multiplication in the Fourier-domain, so the Rayleigh integral of the synthesized field on the reference line y_{ref} can be written in the k_x domain as the product of synthesis operators and the Green's function:

$$\tilde{P}(k_x, y_{\mathrm{ref}}) = \tilde{Q}(k_x) \tilde{G}(k_x, y_{\mathrm{ref}}). \tag{8}$$

From this it is obvious that if the spectrum of the virtual source sound field is known on the reference line, the driving signals in the wave number domain can be calculated with a simple division. The approach is called spectral division method (SDM), as suggested by Ahrens et al. in [1]. The synthesis operator or *driving function* can be calculated in the spatial domain as

$$Q(x) = \mathcal{F}^{-1} \left(\frac{\tilde{P}(k_x, y_{\mathrm{ref}})}{\tilde{G}(k_x, y_{\mathrm{ref}})} \right). \tag{9}$$

The greatest benefit of the approach is that – so far it does not apply any approximation – the synthesis operator holds also in the near field of the secondary sources, even in the low frequency region. The drawback of the technique is that the inverse transform of the spectral ratio rarely can be calculated analytically. The SDM synthesis operator for monochromatic plane wave was derived in [1] and the near-equivalence with traditional WFS formulation was shown, but in the aspect of the present study the comparison of WFS and SDM driving functions for a virtual monopole has greater importance, therefore this will be studied in the following section.

2.2 Comparision of WFS and SDM driving functions for a virtual monopole

To derive the monopole driving function with SDM, the Fourier-transform of three-dimensional Green-function, given by (2) with respect to x is needed in the synthesis plane. The Fourier integral can be calculated by applying Euler's theorem and integral identities [15, 17]. The formula, given in [1] can be simplified using [18] and [14] in order to get a closed form, and finally the Fourier transform reads

$$\tilde{G}(k_x, y) = \int_{-\infty}^{\infty} \frac{1}{4\pi} \frac{\mathrm{e}^{-\mathrm{j}k\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}} \mathrm{e}^{\mathrm{j}k_x x} \mathrm{d}x = -\frac{j}{4} H_0^{(2)} \left(\sqrt{\left(\frac{\omega}{c}\right)^2 - k_x^2} |y| \right). \tag{10}$$

Where $H_0^{(2)}$ is the zeroth-order Hankel function of the second kind.

Using this the driving function of the secondary monopole distribution along $\mathbf{x}_0 = [x\ 0\ 0]^{\mathrm{T}}$ that synthesize the sound field of a monopole at $\mathbf{x}_s = [x_s\ y_s\ 0]^{\mathrm{T}}$ can be derived in the wave number domain. One can apply the Fourier shift theorem to account for the position x_s of the virtual source. The resulting driving function, with respect to reference line y_{ref} reads:

$$\tilde{Q}(k_x) = \frac{\mathrm{e}^{\mathrm{j}k_x x_s} H_0^{(2)} \left(\sqrt{\left(\frac{\omega}{c}\right)^2 - k_x^2} \left(|y_{\mathrm{ref}}| + |y_s|\right) \right)}{H_0^{(2)} \left(\sqrt{\left(\frac{\omega}{c}\right)^2 - k_x^2} |y_{\mathrm{ref}}| \right)}. \tag{11}$$

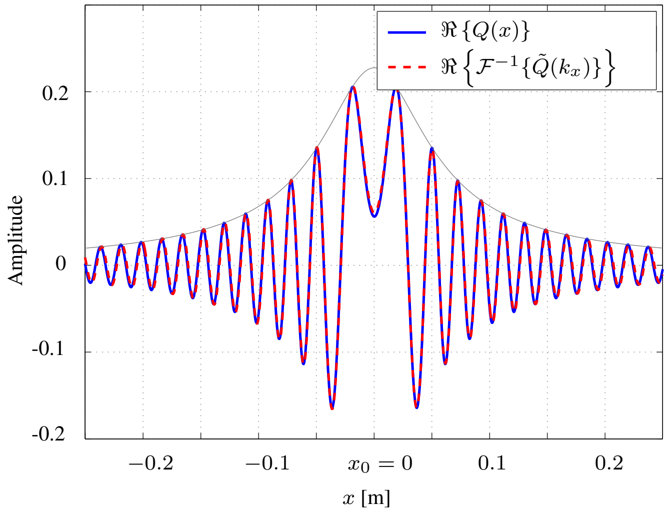

Figure 2: Real part of driving functions calculated in x (blue solid line) and k_x (red dashed line) domain with their absolute value (grey solid line)

By using the large argument asymptotic expansion of the Hankel function [14, 15] the division can be carried out:

$$\tilde{Q}(k_x) = \mathrm{e}^{\mathrm{j}k_x x_s}\sqrt{\frac{|y_{\mathrm{ref}}|}{|y_s| + |y_{\mathrm{ref}}|}}\mathrm{e}^{-\mathrm{j}\sqrt{\left(\frac{\omega}{c}\right)^2 - k_x^2}|y_s|}. \tag{12}$$

The large argument approximation is valid if the argument changes rapidly, therefore y_{s} is a large number, thus the approximation is again valid in the far-field or high frequency region, similarly to the approximation used in spatial domain.

Due to the similar approximations it is reasonable to compare the driving functions derived in x and k_x domain. During the research the inverse Fourier transform of driving function spectrum was not carried out analytically, but a comparison was evaluated numerically. The result is shown in figure 2. It can be seen, that the Rayleigh integral formulation given in (3) and the inverse transform of the spectral ratio perfectly match, so the following general equivalence holds:

$$Q_{2.5\mathrm{D}}(x) = \sqrt{\frac{\mathrm{j}k}{2\pi}}\sqrt{\frac{|y_{\mathrm{ref}}|}{|y_{\mathrm{ref}}| + |y_s|}}\cos\varphi\frac{\mathrm{e}^{-\mathrm{j}kr}}{\sqrt{r}} = \mathcal{F}^{-1}\left\{\mathrm{e}^{\mathrm{j}k_x x_s}\sqrt{\frac{|y_{\mathrm{ref}}|}{|y_s| + |y_{\mathrm{ref}}|}}\mathrm{e}^{-\mathrm{j}\sqrt{\left(\frac{\omega}{c}\right)^2 - k_x^2}|y_s|}\right\}, \tag{13}$$

where $r = \sqrt{(x - x_s)^2 + y_s^2}$ and $\cos\varphi = y_s/r$. The equivalence of the driving functions derived in space and wave number domain makes it easy to demonstrate the effects of linear filtering of these functions, as it will be shown in the next section.

3 Linear filtering of the synthesis operators

In this section we will discuss, how the linear filtering of the previously introduced driving functions can be interpreted and how it influences the synthesized sound field. First, we will discuss how the field of an extended source can be written analytically, then it will be shown that linear filtering of the synthesis operators is equivalent to spatially extending the secondary source elements, or the virtual source.

3.1 Description of spatially extended source distributions

Suppose a one-dimensional spatially extended source distribution along the x axis in the position $\mathbf{x}_{\mathrm{s}} = [x\ y_{\mathrm{s}}\ 0]^{\mathrm{T}}$. According to [3] the pressure field, generated by an extended source in the space may be approximated with this line source distribution in the synthesis plane with proper projection transform, therefore the theory, described below is not restricted to synthesizing virtual sources with only one-dimensional extension.

The radiator does not need to be homogeneous, the strength distribution function is given by the normal velocity on the source distribution $V_{\mathrm{n}}(\mathbf{x}_s)$. The sound field of the extended source can be written with the convolution form of the Rayleigh integral as given in [17]:

$$P(\mathbf{x}) = \mathrm{j}\rho_0 ck V_{\mathrm{n}}(\mathbf{x}_s) \otimes G(\mathbf{x}|\mathbf{x}_s), \tag{14}$$

that reads

$$\tilde{P}(k_x, y) = \mathrm{j}\rho_0 ck \tilde{V}_{\mathrm{n}}(k_x) \tilde{G}(k_x, y - y_{\mathrm{s}}) \tag{15}$$

in the wave number domain.

An important property of the Rayleigh integral is that in the far field of the radiator, where convolution may be approximated by multiplication, the far field directivity pattern of a spatially extended source distribution can be modelled as a monopole, radiating with $D(\varphi)$ [17]:

$$P(\mathbf{x}) = D(\varphi) \frac{\mathrm{e}^{-\mathrm{j}k|\mathbf{x}-\mathbf{x}_s|}}{|\mathbf{x}-\mathbf{x}_s|}, \tag{16}$$

where the directivity function is defined by the Fourier-transform of the source distribution velocity:

$$D(\varphi) = -\frac{\mathrm{j}\rho_0 ck}{2\pi} \tilde{V}_{\mathrm{n}}(k_x) \tag{17}$$

written in the synthesis plane.

3.2 Modifying the synthesis operators

Now we can examine the effect of linear filtering of the synthesis operators. Let's assume a linear, time- and space-invariant filter, defined at a given frequency in the spatial domain by its impulse response $h(x)$, and in the wave number domain by its transfer function $\tilde{H}(k_x)$. Filtering the driving function $Q_{2.5\mathrm{D}}(x)$, given at (4) is given by the convolution with the filter impulse response:

$$Q'(x) = h(x) \otimes Q_{2.5\mathrm{D}}(x) \tag{18}$$

$$\tilde{Q}'(k_x) = \tilde{H}(k_x) \tilde{Q}_{2.5\mathrm{D}}(k_x). \tag{19}$$

The synthesized sound field on the reference line will be the convolution of the filtered driving function and the Green's function:

$$P'(x, y_{\mathrm{ref}}) = Q'(x) \otimes G(x, y_{\mathrm{ref}}) = (h(x) \otimes Q_{2.5\mathrm{D}}(x)) \otimes G(x, y_{\mathrm{ref}}) \tag{20}$$

$$\tilde{P}'(k_x) = \left(\tilde{H}(k_x)\tilde{Q}_{2.5\mathrm{D}}(k_x)\right) \tilde{G}(k_x, y_{\mathrm{ref}}) = \tilde{Q}_{2.5\mathrm{D}}(k_x) \left(\underbrace{\tilde{H}(k_x))}_{\mathrm{j}\rho_0 ck\tilde{V}_{\mathrm{n}}(k_x)} \tilde{G}(k_x, y_{\mathrm{ref}}) \right). \tag{21}$$

From this it is clear that the linear filtering of the driving functions is equivalent to the filtering of the Green's function. This can be interpreted as spatially extending the secondary source elements and its extension will be described by the impulse response of the filter applied. In the far-field region it can be regarded as employing directive monopole secondary sources, and the directivity function is described by the transfer function of the linear filter.

Now assume that we want to synthesize the sound field of a spatially extended source, or a directive monopole in its far field. Traditional WFS operator employed the possibility of synthesizing monopole with given directivity function $D(\varphi)$, so that the synthesis operators read $Q'(x) = D(\varphi)\big|_{\mathbf{x}_0} Q_{2.5\mathrm{D}}(x)$. In the derivation the angular derivative of directivity function was left, therefore the traditional directive operator employed a far field approximation for the directivity function as well.

As we could see from (15) an extended source with the velocity distribution $V_{\mathrm{n}}(\mathbf{x}_s)$ produces a wave field on the reference line y_{ref} that has a spectrum

$$\tilde{P}(k_x, y_{\mathrm{ref}}) = \mathrm{j}\rho_0 ck\tilde{V}_{\mathrm{n}}(k_x)\tilde{G}(k_x, y_{\mathrm{ref}} - y_s). \tag{22}$$

Applying SDM to synthesize this sound field the driving function is written as

$$\tilde{Q}'(k_x) = \frac{\mathrm{j}\rho_0 ck\tilde{V}_{\mathrm{n}}(k_x)\tilde{G}(k_x, y_{\mathrm{ref}} - y_s)}{\tilde{G}(k_x, y_{\mathrm{ref}})}. \tag{23}$$

In the previous section it was shown, that the part $\tilde{G}(k_x, y_{\mathrm{ref}} - y_s)/\tilde{G}(k_x, y_{\mathrm{ref}})$ equals exactly to the monopole driving function $\tilde{Q}_{2.5\mathrm{D}}$ both in wave number and spatial domain, given by (12) and (4). Equation (23) therefore can be rewritten as

$$\tilde{Q}'(k_x) = \mathrm{j}\rho_0 ck\tilde{V}_{\mathrm{n}}(k_x)\tilde{Q}_{2.5\mathrm{D}}(k_x). \tag{24}$$

That means synthesizing the sound field of the virtual source distribution can be done by filtering the monopole operator with a linear filter, that have a transfer function $\tilde{H}(k_x) = \mathrm{j}\rho_0 ck\tilde{V}_{\mathrm{n}}(k_x)$ and has an impulse response, that equals to the spatial distribution of the extended source to synthesize.

As a conclusion we can say that linear filtering of the synthesis operators in case of linear, continuous secondary source distribution has two interpretations (up to a constant): to spatially extend the elements of the secondary source distribution, or to spatially extend the virtual monopole source by the impulse response of the applied filter. In other words, the virtual and secondary source extensions (and so their spectra) are interchangeable. Because the radiation pattern is an approximation of the radiation spectrum, therefore the virtual and secondary source directivity are also interchangeable.

4 Applications of linear filtering

In this section we will see how theory – shown previously – can be applied to different synthesis problems. Two problems and their solution will be discussed. The first and most obvious application is the synthesis of a field generated by a directive source. After that we will see, how spatial aliasing effects may be decreased with proper filtering of the driving function.

4.1 Synthesizing non-omnidirectional sources

As we could see in the previous section, filtering the driving functions is equivalent to spatially extending the virtual sound source. This will radiate with a radiation pattern, specified by the transfer function of the applied filter. In the far-field of the virtual source distribution it can be modelled as a directive monopole. This gives us the possibility to synthesize the sound field of a non-omnidirectional point source with given directivity. In the followings the synthesis of such a point source will be demonstrated through a concrete example.

Let's assume a point source positioned at $\mathbf{x}_{\mathrm{s}}$ orientated towards the secondary source line. The source radiates with a predefined directivity function $D(\varphi)$. In this example let this directivity function be

$$D(\varphi) = \cos(2N\varphi), \tag{25}$$

where $N \in \mathbb{Z}$. The given directivity describes a multipole, having $4, 6, 8...$ lobes.

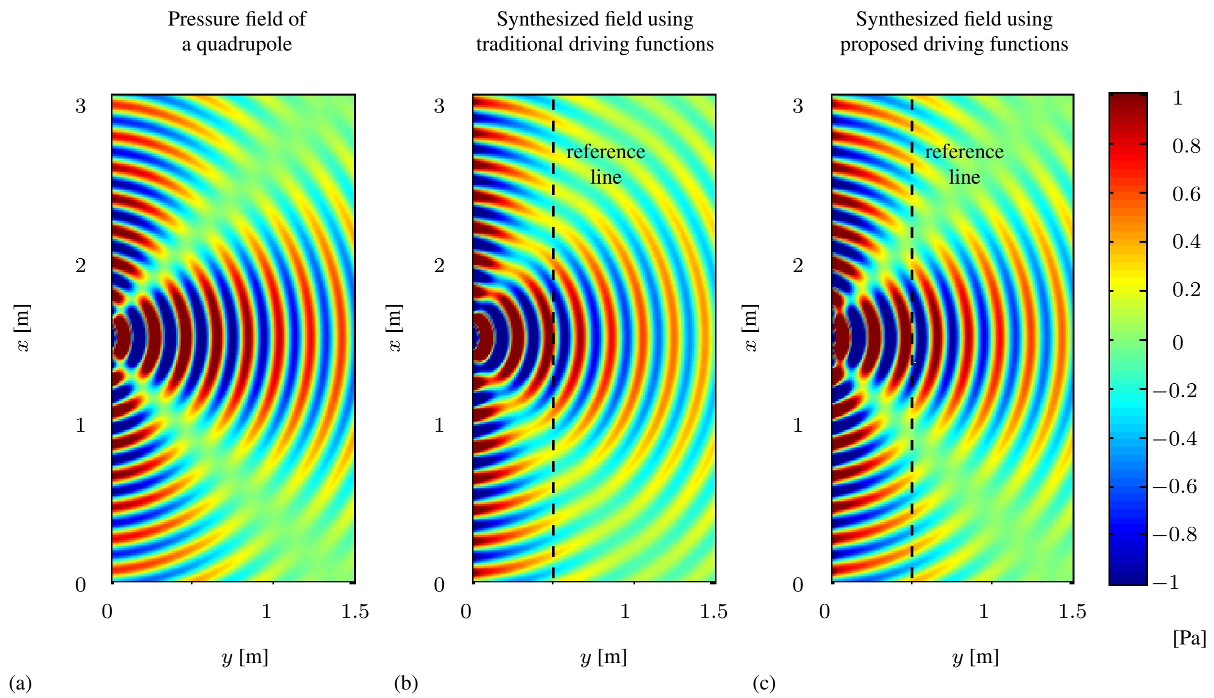

Figure 3: The field, generated by an acoustic quadrupole (a), and the synthesized sound field of the quadrupole with the traditional synthesis operators (b) and the proposed method (c)

The angle dependent far-field directivity function and the spectrum of the radiated field examined on the reference line are strongly related, as $\varphi = \arcsin \frac{k_x}{k}$. The spectrum of the radiation thus can be written directly from the directivity function (omitting $\mathrm{j}\rho_0 ck$):

$$\tilde{H}(k_x) = D\left(\arcsin \frac{k_x}{k}\right) = \cos\left(2N \arcsin \frac{k_x}{k}\right). \tag{26}$$

In order to synthesize the pressure field generated by this source, we have to filter the traditional monopole driving functions with a transfer function, given by (26), so the resulting operator in the k_x-domain is written as

$$\tilde{Q}_{\text{direct}}(k_x) = \tilde{H}(k_x)\tilde{Q}_{2.5\text{D}}(k_x), \tag{27}$$

where $\tilde{Q}_{2.5\text{D}}(k_x) = \mathcal{F}\left(Q_{2.5\text{D}}(x)\right)$ is the k_x form of the traditional monopole driving functions.

In figures 3(b) and (c) the result of the synthesis can be seen in the case at $N = 2$, which can be interpreted as the synthesis of the field, generated by an acoustic quadrupole. The original field is shown in 3(a). The virtual quadrupole is located at $\mathbf{x}_S = [-0.1, 1.5, 0]^{\mathrm{T}}$ (m) and the secondary sources are on the line $\mathbf{x}_0 = [x\ 0\ 0]^{\mathrm{T}}$, while the reference line is set to $y_{\text{ref}} = 0.5$ m.

In the traditional way the driving function is calculated as given by [3]:

$$Q_{\text{quad}}(x) = \cos\left(4\varphi\right) Q_{2.5\text{D}}(x). \tag{28}$$

The synthesized field using the traditional driving functions can be seen in figure 3(b). In the traditional derivation of (28) the angular derivative of the directivity function is omitted, which approximation is valid only in the far-field, therefore a virtual source, close to the secondary line can be synthesized only with significant deviations. These effects are clearly visible in figure 3(b) on the rapidly changing parts of the directivity functions, which are located around the zero-crossings, close to the virtual source.

The proposed driving functions are calculated as

$$Q_{\text{quad}}(x)' = \mathcal{F}^{-1}\left(\cos\left(4\arcsin\frac{k_x}{k}\right)\tilde{Q}_{2.5\text{D}}(k_x)\right). \tag{29}$$

In figure 3(c) the result of synthesis can be seen, using this proposed driving function. In contrast to the traditional way, the proposed method applies no approximation with respect to the directivity function, therefore even the nearfield of the virtual source can be synthesized correctly. The proposed method thus can be regarded as the perfect, approximation-free derivation of the synthesis operators for directive sources.

To get more insight how linear filtering effects the driving functions, it's expedient to examine the operation in the spatial domain as well. The impulse response is calculated as the inverse Fourier-transform of transfer function (26). Equation (26) can be transformed analytically for even N (which explains our initial choice of $2N$) through the Chebyshev polynomials:

$$\tilde{H}(k_x) = \cos\left(4\arcsin\frac{k_x}{k}\right) = 1 - \frac{2k_x^2}{k^2}, \tag{30}$$

$$h(x) = \mathcal{F}\left(\tilde{H}(k_x)\right) = \delta(x) - \frac{2\delta''(x)}{k^2}. \tag{31}$$

This impulse response describes the spatial extension of the virtual source. The result can be understood as the distribution function of a quadrupole along the x-axis, which is indeed, defined through the derivatives of the Dirac function.

It's important to note, that spatial extension, described by $h(x)$ contains only the zeroth and second derivatives of the Dirac-delta function. According to the properties of the Dirac-function, the linear combination of its n-th order derivatives still contains components only in the origin and the support of the function converges to zero. That means, that the virtual source, synthesized by (27) is truly a point source, radiating with a cosine directivity function. During the synthesis therefore we don't approximate the directive point source with a extended source distribution but we directly synthesize the field of the directive point source without any far-field approximation.

We can write the resulting driving function in spatial domain as the convolution of the filter impulse response with the monopole synthesis operator: using that $\delta(x) \otimes f(x) = f(x)$ and $\delta''(x) \otimes f(x) = f''(x)$ the convolution can be carried out analytically:

$$Q_{\text{quad}}(x) = h(x) \otimes Q_{2.5\text{D}}(x) = Q_{2.5\text{D}}(x) - \frac{2}{k^2}\frac{\partial^2 Q_{2.5\text{D}}(x)}{\partial x^2}. \tag{32}$$

The derivation may be calculated, so a direct formula can be found, but further investigation of these effects are not topic of the present paper.

4.2 Avoiding effects of spatial sampling

So far we were dealing with continuous secondary source distributions. In practice it's not a realizable condition, the effects of discrete secondary sources must be taken into consideration.

The substitution of continuous source line with elements of secondary sources can be regarded as a spatial sampling process. Sampling is modelled as the multiplication of the continuous function with a series of Dirac functions called sampling function, or Dirac comb. Here the position of Dirac pulses indicates the position of the theoretical secondary monopoles, therefore the Dirac delta series represents the spatial distribution function of the secondary sources applied. In case of equidistant secondary sources the sampled driving function in spatial domain is written as

$$Q_{\text{s}}(x) = Q(x)\frac{1}{dx}\sum_{n=-\infty}^{\infty}\delta(x - ndx) = Q(x)\Delta_{dx}(x), \tag{33}$$

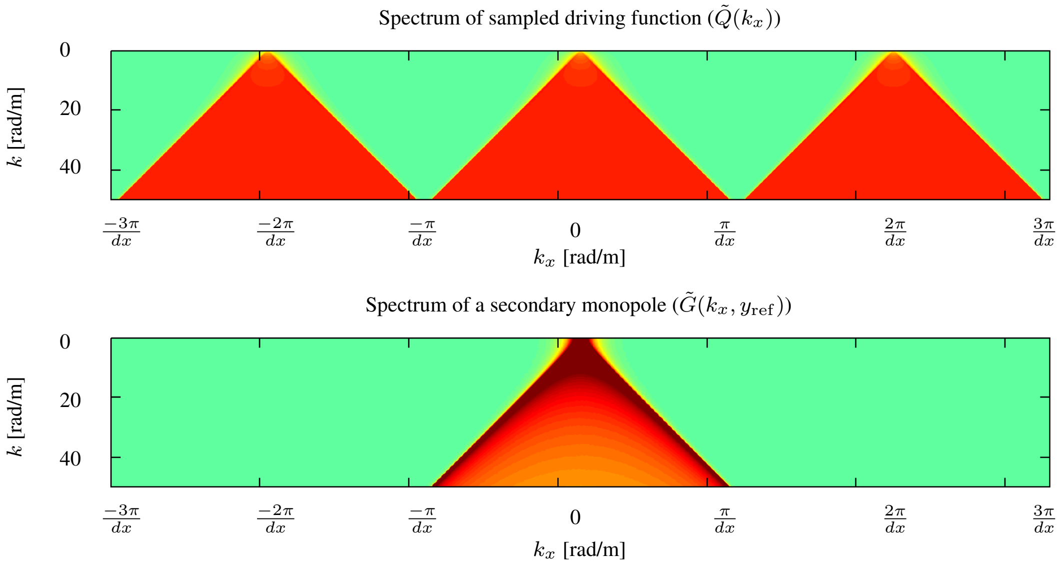

Figure 4: Spectrum of the sampled driving function and the Green's function. The synthesized sound field is the product of the two spectra: $\tilde{P}(k_x, y_{\text{ref}}) = \tilde{Q}(k_x)\tilde{G}(k_x, y_{\text{ref}})$

where dx is the sampling distance. The spectrum of sampling function is a Dirac series itself, so in the k_x domain the spectrum of the sampled driving function will be the superposition of the original spectrum, shifted along k_x by $n\frac{2\pi}{dx}$:

$$\tilde{Q}_{\text{s}}(k_x) = \tilde{Q}(k_x) \otimes \tilde{\Delta}_{dx}(k_x) = \sum_{n=-\infty}^{\infty} \tilde{Q}\left(k_x - n\frac{2\pi}{dx}\right). \tag{34}$$

The wave field reproduced by the sampled driving function can be written on the reference line as

$$\tilde{P}(k_x, y_{\text{ref}}) = \tilde{G}(k_x, y_{\text{ref}})\left(\tilde{Q}(k_x) \otimes \tilde{\Delta}_{dx}(k_x)\right). \tag{35}$$

As the spectrum of the Green's function is bandlimited to the propagation domain the secondary source distribution can be regarded as interpolation filter during the reproduction.

The spectrum of the sampled driving function for a virtual monopole and the spectrum of the applied secondary monopole can be seen in figure 4. There are two conditions to avoid the artifacts due to the secondary source sampling: on one hand, the spectrum of the continuous driving function must be bandlimited to $k_{x,\text{Nyq}} = \pi/dx$, so that the shifted spectra of driving function are not overlapped, therefore no spatial aliasing occurs. On the other hand the spectrum of secondary source elements, used for reproduction (in practice the angular spectrum of loudspeaker applied) must be also bandlimited to $k_{x,\text{Nyq}}$ to reproduce only the baseband components of the original sound field, which means interpolation in the spatial domain.
The latter condition requires that the secondary sources do not radiate in lateral directions on high frequencies, which requirement is fulfilled under certain assumptions for spatially extended sources, like a dynamic loudspeaker. However, with prefiltering the driving function one has greater freedom for optimization.

Assume that both anti-aliasing and reconstruction filter attenuates perfectly in the stop-band, therefore sampling artifacts do not appear in the reproduced field. In this case the synthesized sound field can be written by using only the baseband components:

$$\tilde{P}(k_x, y_{\text{ref}}) = \left(\tilde{H}_{\text{a.a}}(k_x)\tilde{Q}(k_x)\right)\left(\tilde{G}(k_x, y_{\text{ref}})\tilde{H}_{\text{rec}}(k_x)\right) = \left(\tilde{H}_{\text{a.a}}(k_x)\tilde{H}_{\text{rec}}(k_x)\tilde{Q}(k_x)\right)\tilde{G}(k_x, y_{\text{ref}}) \tag{36}$$

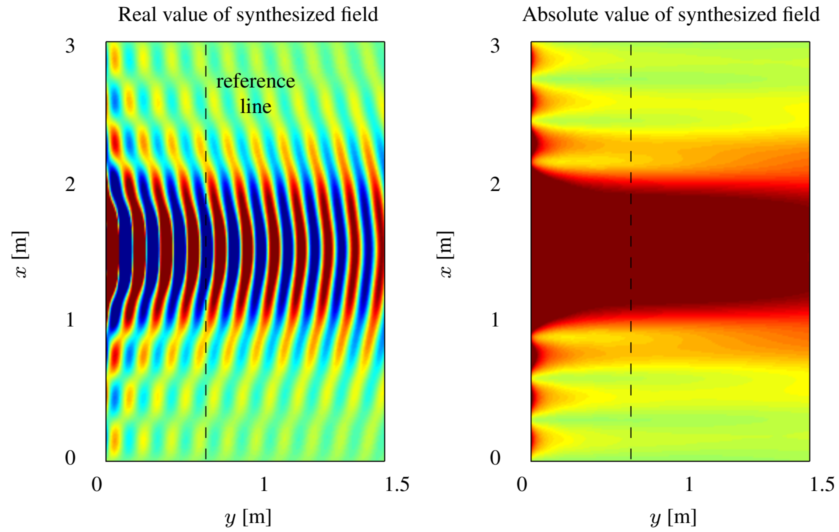

Figure 5: Real value and absolute value of the synthesized field with sampled driving function, applying ideal low-pass filter for anti-aliasing and simply modelled loudspeaker for reproduction

where $\tilde{H}_{\text{a.a}}(k_x)$ and $\tilde{H}_{\text{rec}}(k_x)$ are the transfer functions of anti-aliasing and reconstruction filters. The spectrum of the synthesized source will be determined by the product of the anti-aliasing and reconstruction filter, while the spatial extension will be described by the convolution of their impulse responses. In practice, the properties of reconstruction filter are given by the loudspeakers, applied. If it is known accurately, one can design the anti-aliasing filter to avoid spatial aliasing and to set a desired virtual source directivity.

The main aim of optimization, when anti-aliasing filter is chosen is to bandlimit the driving function to $k_{x,\text{Nyq}} = \pi/dx$, without modifying the relevant properties of the original virtual source. Assume that we want to synthesize the field of a virtual monopole, having a Dirac-delta distribution function. Obviously with a sampled source distribution we are not able to synthesize this field perfectly. Our aim is to approximate the Dirac-delta with $h_{\text{a.a}} \otimes h_{\text{rep}}$ resultant filter impulse response, beside satisfactory stop-band attenuation.

In literature on the current topic (e.g [4, 5]) low-pass filtering was done by setting $|k_x| > k_{x,\text{Nyq}}$ components of the driving function to zero:

$$H_{\text{a.a}}(k_x) = \sqcap(\frac{k_x}{2k_{x,\text{Nyq}}}) = \begin{cases} 0 & \text{if } |k_x| > k_{x,\text{Nyq}} \\ \frac{1}{2} & \text{if } |k_x| = k_{x,\text{Nyq}} \\ 1 & \text{if } |k_x| < k_{x,\text{Nyq}} \end{cases} \tag{37}$$

$$h_{\text{a.a}}(x) = \frac{2\pi}{dx}\text{sinc}\left(\frac{2\pi x}{dx}\right). \tag{38}$$

If the virtual source is far enough from the secondary source distribution, the generated wave field tends to be a plane wave, the driving function has a constant amplitude in the spatial domain, and its spectrum is a Dirac-delta. If it's under the Nyquist-wave number it remains unchanged after filtering, so the constant amplitude oscillation is invariant to the convolution with a sinc function. However, if the source is close to the secondary source distribution, then the x domain function tends to a Dirac-delta. If we filter it with the ideal low-pass filter we extend the monopole into a source distribution, described by the sinc function along the x axis.

The same filter for anti-aliasing and reproduction is not a reasonable presumption in this case, as it would require infinite secondary source elements. To investigate the effect of such an anti-aliasing filter a simple loudspeaker model was made. To model the loudspeaker a baffled piston is a good approximation, which is modelled in the x domain as $h_{\text{rec}}(x) = 2\sqrt{r^2 - x^2}$, and its transfer function is given by $\tilde{H}_{\text{rec}}(k_x) =$

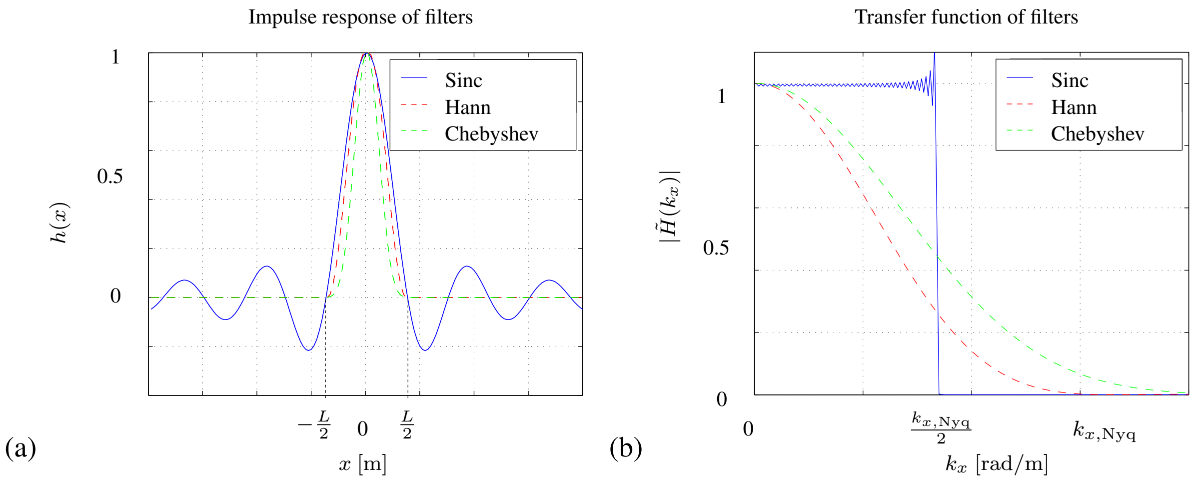

Figure 6: Impulse response (a) and absolute value of transfer function (b) of the ideal low-pass, raised cosine (Hann) and Dolph-Chebyshev filters

$J_1(\frac{k_x r}{2})/k_x r$ [3]. The impulse response of the resultant filter results in an infinite, "sinc-like" virtual source extension, instead of the original virtual monopole.

The effects, described so far are demonstrated in figure 5. In this example a virtual monopole, radiating at $f = 1.8$ kHz ($k \approx 33$ rad/m) is placed at $\mathbf{x}_\mathrm{s} = [-0.5\ 1.5\ 0]^\mathrm{T}$ m. The secondary source is sampled with a sampling distance $dx = 0.15$ m, therefore $k_{x,\mathrm{Nyq}} \approx 21$ rad/m, which is the cut-off wave number of the ideal low-pass filter. The reconstruction filter is modelled as it's explained above. In the figures the real and absolute value of the synthesized field can be observed. It is clearly visible, that the virtual source has an infinite extension, described by the sinc function. This means that this simple approach of low-pass filtering is not a passable method to avoid spatial aliasing.

To avoid this phenomena we would need a filter with both finite impulse response and finite transfer function. The impulse response must be as narrow as possible, while there should be a sharp transition above the cut-off wave number in the wave number domain. These are not physically possible conditions, however the considerations lead us to the application of window-functions for lo-pass filtering. During the research the application of raised cosine (Hann) and Chebysev window was investigated.

The impulse response and transfer function of the raised cosine window, with a length L is given as:

$$h_\mathrm{Hann}(x) = \frac{1}{2}\left(1 - \cos\left(2\pi \frac{x}{L}\right)\right), \mathrm{if} x < L \tag{39}$$

$$\tilde{H}_\mathrm{Hann}(k_x) = \frac{|L|}{\sqrt{32\pi}}\left(2\mathrm{sinc}\left(\frac{k_x L}{2}\right) - \mathrm{sinc}\left(\frac{k_x L}{2} - \pi\right) - \mathrm{sinc}\left(\frac{k_x L}{2} + \pi\right)\right). \tag{40}$$

The sum of sinc functions forms a low-pass filter with high side-lobe attenuation, that's cut-off wave number can be optimized. The transfer function of the designed filter can be seen in figure 6(b). The filter was designed so that it has a zero on the sampling wave number, and it's cut-off wave number is approximately the Nyquist wave-number.

If we assume that the anti-aliasing and reconstruction filter are the same, omitting the aliasing effects the spatial extension of the virtual source, synthesized with the linearly filtered driving function is characterised by the self-convolution of the impulse response (39), which is a raised cosine itself.

$$h(x)_\mathrm{a.a.} \otimes h(x)_\mathrm{rec} = \frac{1}{4}(\pi - 2)\left(1 - \cos 2\pi\frac{x}{L}\right), \mathrm{if} x < L. \tag{41}$$

It means, that the virtual source will be a a finite source distribution, described by one lobe of a raised cosine function, which is a far better approximation of a monopole than the previously examined sinc square function. The virtual source distribution (the filter impulse response) can be seen in figure 6(a).

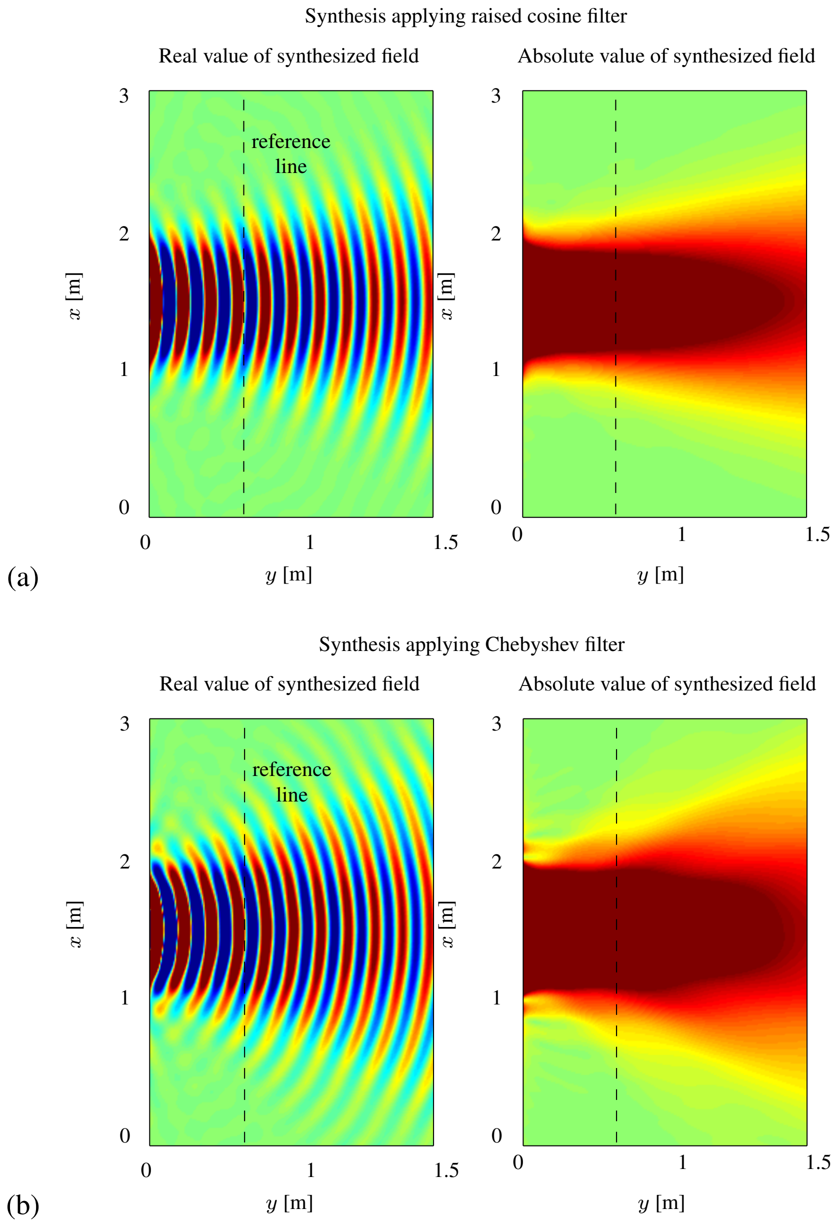

Figure 7: The real value and absolute value of a reproduced sound field of a monopole, applying raised cosine (a), and Chebyshev (b) filters for anti-aliasing and reproduction

Beside the raised cosine, the application of Dolph-Chebyshev-window was also investigated, which was first used for antenna design, to achieve a narrow main-lobe pattern, while simultaneously restricting side-lobe response, therefore it had a very similar use to the present topic. The window is designed so that the side lobe magnitude is given dB below the main lobe magnitude so the design parameter is the passband-stopband ratio, $10 \log \frac{H_{\mathrm{PB}}}{H_{\mathrm{SB}}}$. Of course, the lower the side lobe level, the wider the main lobe will be. The window has a rather complex impulse response and transfer function, it can be found in e.g. [19]. During the research a Chebyshev-window with $10 \log \frac{H_{\mathrm{PB}}}{H_{\mathrm{SB}}} = 100$ dB was used, having approximately zero magnitude on sampling wave number.

The impulse responses and the transfer functions of the filters can be seen in figures 6(a) and (b). The Chebyshev-window has slightly narrower impulse response, but it has a lower attenuation above the Nyquist wave number, therefore aliasing effects in the synthesized field will be present.

The real and absolute value of the synthesized sound fields can be seen in figures 7, employing raised cosine (a) and Chebyshev filters (b). In both cases, the area of correct reproduction significantly narrows, however, in the radiated areas the reproduction of the virtual monopole is possible. As the passband of the Chebyshev filter is wider, than that of Hann window, the area of reproduction is larger, but in latter case aliasing effects appear in the synthesized wave field. As conclusion we can say that for a given sampling distance the area of correct reproduction can be increased on the cost of involving aliasing components into synthesis.

5 Conclusion

This paper presented a throughout treatise on the effects of linear filtering of the WFS driving functions. The analysis was carried out in the spatial and wave number domains. It was shown, that in the far-field of the synthesis line the resultant driving functions can be considered to be equal, therefore the two methods can be treated as the parallel solution of the same problem in spatial and spectral domains.

Using these results the effects of linear filtering of the monopole synthesis operators were investigated. It was shown, that linear filtering is equivalent to spatially extending the secondary source elements, and using far-field approximations, it is also equal to spatially extending the virtual source. Therefore, in the case of a continuous linear secondary source distribution, the virtual and secondary source properties are inversely interchangeable.

Finally two applications of linear filtering were shown. First, filtering was directly used to synthesize the sound field generated by a directive point source. It turned out that point sources having multipole directivity can be be synthesized without any approximation. As sources with complex radiation pattern can be expanded into series of base functions described by multipole directivity functions, the technique shown below may be further improved to synthesize sources with more complex radiation patterns.
Secondly, the proper filter design for anti-aliasing filtering was presented. To avoid aliasing effects anti-aliasing filtering is needed. In the last chapter the odds of applying ideal low-pass filter were exhibited. It could be seen, that to preserve the original properties of the virtual source applying a Hann or Chebysev-filter gives better results in front of the virtual source, however, the area of correct reproduction will decrease. This area can be increased in the cost of involving aliasing components into the synthesized sound field. The audibility of the spatial aliasing remains an open question: if the amount of permitted spatial aliasing is known, then the anti-aliasing filter can be designed so that the area of reproduction is maximized besides minimal audible aliasing artifacts.

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