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AN EXTENSION OF THE VALIDITY OF THE STATIONARY PHASE FORMULA*

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1. Introduction. The method of stationary phase is applied to integrals of the form

$$I(k) = \int e^{ik\varphi(x)} g(x)\, dx. \tag{1.1}$$

An isolated stationary point is defined as a point $\tilde{x}$ such that $\varphi'(\tilde{x}) = 0$ and $\varphi''(\tilde{x}) \neq 0$. If $\tilde{x}$ is the only critical point in the interval of integration, then I has an asymptotic expansion (valid for large positive k) of the form

$$I(k) \sim \frac{e^{ik\varphi(\tilde{x})}}{\sqrt{k}} \sum_{j=0}^{\infty} \frac{a_j}{k^j}, \tag{1.2}$$

where a_j involves the values of g and its derivatives up to order $2j$ at $\tilde{x}$ and derivatives of φ up to order $2j + 2$ at $\tilde{x}$. Such an expansion is not valid if $\varphi''(\tilde{x}) = 0$; indeed, each coefficient a_j is infinite in such a case. On the other hand, experience with certain uniform expansions (see Chester, Friedman and Ursell [1]) shows that expansions of the form (1.2) are valid even for small values of $\varphi''(\tilde{x})$, provided that $k^{1/3}\varphi''(\tilde{x})$ is large. For example, the Debye approximation

$$J_k(kr) \sim \left(\frac{2}{\pi k\sqrt{r^2 - 1}}\right)^{1/2} \cos\left[k\sqrt{r^2 - 1} - \operatorname{arc\,cos}\frac{1}{r} - \frac{\pi}{4}\right] \quad \text{if} \quad r > 1 \tag{1.3}$$

is obtained by applying the method of stationary phase to an integral representation of the Bessel function and retaining only the first term of the expansion. We may ask how small $r - 1$ must be before (1.3) fails. The uniform asymptotic expansion of $J_k(kr)$ shows that (1.3) is a valid first approximation provided that $k^{1/3}\sqrt{r - 1}$ is large.

In the general case of (1.1), φ and g (and hence $\tilde{x}$) might depend upon a parameter λ (the analogue of $\sqrt{r - 1}$) in such a way that $\varphi''(\tilde{x}, \lambda) \to 0$ as $\lambda \to 0$. (We continue to denote derivatives of φ with respect to x by a prime.) If λ is restricted (depending upon k) in such a way that $k^{1/3}\varphi''(\tilde{x}, \lambda)$

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is large, then $I(k, \lambda)$ may be represented by

$$I(k,\lambda) \sim \frac{e^{ik\varphi(\tilde{x},\lambda)}}{k^{1/3}\sigma^{1/2}} \sum_{j=0}^{\infty} \frac{b_j(\lambda)}{\sigma^{3j}}, \tag{1.4}$$

where

$$\sigma = k^{1/3} \left| \frac{1}{2}\varphi''(\tilde{x}, \lambda) \right|. \tag{1.5}$$

The coefficients b_j involve the values of g and its derivatives of order up to $2j$ at $\tilde{x}$ and derivatives of φ up to order $2j + 2$ at $\tilde{x}$. Of course, (1.4) is valid wherever (1.2) is valid, but (1.4) provides a transition between the situation where the jth term in the expansion has order $k^{-j+1/2}$ (when σ has order $k^{1/3}$) and the situation where all terms in the expansion have nearly the same order in k (where σ nearly has the order of k^0). If σ has the order of k^0, then neither (1.2) nor (1.3) is valid. We also show that the expansion (1.4) is contributed by an interval around $\tilde{x}$ whose length has order $k^{-1/3}$. It should be emphasized that the expansion (1.4) actually refers to a different situation than (1.2), due to the presence of the parameter λ. The range of values of λ for which (1.4) is valid is larger than for (1.2), but it is dependent upon k.

A similar procedure can be applied to an integral of the form (1.1) which involves an endpoint of the interval of integration. If x_0 is such an endpoint, if g has a factor $(x - x_0)^r$, and if $\varphi'(x_0) \neq 0$, then the corresponding contribution to the expansion of I has the form

$$I(k) \sim \frac{1}{k^{r+1}} \sum_{j=0}^{\infty} \frac{c_j}{k^j}, \tag{1.6}$$

where c_j involves the values of $(x - x_0)^{-r} g$ and its derivatives up to order j at x_0 and derivatives of φ up to order $j + 1$ at x_0. We permit φ and g to depend upon λ, and we set

$$\tau = k^{1/2} \,|\, \varphi'(x_0, \lambda) \,|. \tag{1.7}$$

If τ is large, we have

$$I(k,\lambda) \sim \frac{1}{k^{(1+r)/2}\tau^{1+r}} \sum_{j=0}^{\infty} \frac{d_j(\lambda)}{\tau^{2j}}, \tag{1.8}$$

where d_j involves the values of $(x - x_0)^{-r} g$ and its derivatives up to order j at x_0 and derivatives of φ up to order $j + 1$ at x_0. Thus our expansion (1.8) extends the validity of the endpoint formula (1.6) to the point where successive terms of the expansion have nearly the same order in k. Our analysis also shows that the expansion (1.8) arises from an interval around x_0 whose length has order $k^{-1/2}$.

Our immediate motivation for the present work was to verify an asymptotic solution of a diffraction problem (see Ludwig [3]). However, the results appear to be useful in a wider context, for example, to justify integration of asymptotic expansions which break down at one or more points.

2. An interior stationary point. The statement of our main theorem is complicated by the need to ensure that there is only one stationary point in the interval of integration. Roughly speaking, we must assume that $\varphi'(x, \lambda)$ is different from zero except when x is near $\tilde{x}$. For our present purposes, "x is near $\tilde{x}$" means that $x - \tilde{x}$ has order less than $k^{-1/3}$ (independent of λ) and "φ' is different from zero" means that q and r, defined by

$$q = \frac{\sigma}{k^{2/3}\varphi'(x, \lambda)}, \tag{2.1}$$

$$r = \frac{\varphi''(x, \lambda)}{k^{1/3}\varphi'(x, \lambda)}, \tag{2.2}$$

are bounded independently of k and λ.

THEOREM 1. *It is assumed that $g(x, \lambda)$ and $\varphi(x, \lambda)$ are infinitely differentiable with respect to x and all derivatives of g and φ are continuous with respect to λ, g has compact support in x and λ, $\varphi'(\tilde{x}, \lambda) = 0$, and there exists a number $B > 0$ (independent of k and λ) such that q and r (defined by (2.1) and (2.2)) are bounded independently of k and λ where k is large and $|(x - \tilde{x})k^{1/3}| > B$. Then for any N,*

$$\begin{aligned} I(k, \lambda) &= \int_{-\infty}^{\infty} e^{ik\varphi(x,\lambda)}\, g(x, \lambda)\, dx \\ &= \frac{e^{ik\varphi(\tilde{x},\lambda)}}{k^{1/3}\sigma^{1/2}} \sum_{j=0}^{N} \frac{b_j(\lambda)}{\sigma^{3j}} + k^{-1/3}O(\sigma^{-3N-3-1/2}), \end{aligned} \tag{2.3}$$

where

$$\sigma = k^{1/3}\,|\tfrac{1}{2}\varphi''(\tilde{x}, \lambda)| \tag{2.4}$$

and b_j involves g and its derivatives up to order $2j$ at $\tilde{x}$ and derivatives of φ up to order $2j + 2$ at $\tilde{x}$.

Remark. The hypothesis of Theorem 1 can be weakened by use of the lemma which appears at the end of this section. There it is shown that there exists a number $B_1(\lambda)$ (independent of k) such that q and r are bounded if $k^{-1/3}B \leqq |x - \tilde{x}| \leqq B_1(\lambda)$. Thus, in order to apply Theorem 1, it is only necessary to bound q and r where $|x - \tilde{x}| \geqq B_1(\lambda)$.

The proof of Theorem 1 consists in breaking the interval of integration in (2.3) into two pieces. In the interval where x is near $\tilde{x}$, a change of parameter from k to σ and a change of independent variable results in an integral to which the usual method of stationary phase is applicable, even if $\varphi''(\tilde{x})$

is not bounded away from zero. The integral over the interval where x is bounded away from $\tilde{x}$ is estimated by an integration by parts. Since λ does not appear explicitly in the calculations, we shall not indicate its presence. However, all functions which appear will depend continuously on λ.

It follows from Taylor's theorem that

$$(2.5)\qquad k\,|\,\varphi(x)-\varphi(\tilde{x})\,| = k^{2/3}\sigma(x-\tilde{x})^2 + k(x-\tilde{x})^3 f(x-\tilde{x})$$

with an appropriate function f. We define y by means of

$$(2.6)\qquad y = k^{1/3}(x-\tilde{x}),$$

and we assume for the moment that y is bounded, i.e., we consider a neighborhood of x of width proportional to $k^{-1/3}$. We define ζ by means of

$$(2.7)\qquad \sigma\zeta^2 = k\,|\,\varphi(x)-\varphi(\tilde{x})\,| = \sigma y^2\left[1+\frac{y}{\sigma}f(k^{-1/3}y)\right].$$

In view of (2.4) we may write

$$(2.8)\qquad f(k^{-1/3}y) = f\left(\frac{|\,\frac{1}{2}\varphi''(\tilde{x})\,|\,y}{\sigma}\right) = f_1\left(\frac{y}{\sigma}\right),$$

and (2.7) may be rewritten as

$$(2.9)\qquad \frac{\zeta}{\sigma} = \frac{y}{\sigma}\left[1+\frac{y}{\sigma}f_1\left(\frac{y}{\sigma}\right)\right]^{1/2}.$$

If ζ is bounded and σ is large enough, we may apply the implicit function theorem to obtain

$$(2.10)\qquad \frac{y}{\sigma} = \frac{\zeta}{\sigma}\left[1+f_2\left(\frac{\zeta}{\sigma}\right)\right]$$

and thus,

$$(2.11)\qquad \frac{dy}{d\zeta} = 1+f_3\left(\frac{\zeta}{\sigma}\right)$$

with appropriate functions f_2 and f_3.

We can find a number A such that $|\,y\,| \geqq B$ if $|\,\zeta\,| \geqq A$, and we choose a C^∞ function p such that

$$(2.12)\qquad p(s) = \begin{cases} 1 & \text{if } s < A^2, \\ 0 & \text{if } s > 2A^2. \end{cases}$$

Then (2.3) can be written as

$$(2.13)\qquad \int e^{ik\varphi} g\,dx = \int e^{ik\varphi} g p(\zeta^2)\,dx + \int e^{ik\varphi} g(1-p(\zeta^2))\,dx = I_1 + I_2.$$

In the support of $p(\zeta^2)$ we have $\zeta^2 < 2A^2$, and hence from (2.10), y is bounded. Using (2.6), (2.7) and (2.11), we obtain

$$I_1 = e^{ik\varphi(x)} \int e^{\pm i\sigma\zeta^2} g(\tilde{x} + k^{-1/3}y) \left(1 + f_3\left(\frac{\zeta}{\sigma}\right)\right) p(\zeta^2) \frac{d\zeta}{k^{1/3}}, \tag{2.14}$$

where $\pm = \operatorname{sgn}(\varphi(x) - \varphi(\tilde{x}))$. Applying Taylor's theorem to the product $g(\tilde{x} + k^{-1/3}y)(1 + f_3(\zeta/\sigma))$, we obtain

$$I_1 = e^{ik\varphi(\tilde{x})} \int e^{\pm i\sigma\zeta^2} \left[\sum_{l=0}^{2N+1} g_l \frac{\zeta^l}{\sigma^l} + \frac{\zeta^{2N+2}}{\sigma^{2N+2}} h\left(\frac{\zeta}{\sigma}\right)\right] p(\zeta^2) \frac{d\zeta}{k^{1/3}}. \tag{2.15}$$

We note that g_l is a linear combination of derivatives of g up to order l at $\tilde{x}$, whose coefficients involve the derivatives of φ up to order $l + 2$ at $\tilde{x}$. The symmetry of the integrand implies that odd powers of ζ do not contribute to the integral. Thus we obtain

$$I_1 = \frac{e^{ik\varphi(\tilde{x})}}{k^{1/3}} \sum_{j=0}^{N} \frac{g_{2j}}{\sigma^{2j}} \int e^{\pm i\sigma\zeta^2} \zeta^{2j} p(\zeta^2)\, d\zeta + I_3, \tag{2.16}$$

where

$$I_3 = \frac{e^{ik\varphi(\tilde{x})}}{k^{1/3}} \int e^{\pm i\sigma\zeta^2} \frac{\zeta^{2N+1}}{\sigma^{N+1/2}} h\left(\frac{\zeta}{\sigma}\right) p(\zeta^2) \frac{d\zeta}{k^{1/3}}. \tag{2.17}$$

Now we can apply the usual method of stationary phase to the integrals in (2.16) and (2.17) (see Erdélyi [2]). The result is

$$I_1 = \frac{e^{ik\varphi(\tilde{x})}}{k^{1/3}\sigma^{1/2}} \sum_{j=0}^{N} \frac{b_j}{\sigma^{3j}} + \frac{1}{k^{1/3}\sigma^{1/2}} O\left(\frac{1}{\sigma^{3N+3}}\right), \tag{2.18}$$

where

$$b_j = \sqrt{\pi}\, j!\, e^{\pm i\pi(j+1/2)/2} g_{2j}. \tag{2.19}$$

Now it only remains to estimate the integral I_2, where $k^{1/3} | x - \tilde{x} | \geqq B$ in the support of $1 - p(\zeta^2)$. We introduce a new variable ψ by means of

$$\sigma\psi = k(\varphi(x) - \varphi(\tilde{x})). \tag{2.20}$$

Then

$$\frac{k^{1/3}}{\psi'(x)} = \frac{\sigma}{k^{2/3}\varphi''(x)} = q, \tag{2.21}$$

$$\frac{\psi''(x)}{k^{1/3}\psi'(x)} = \frac{\varphi''(x)}{k^{1/3}\varphi'(x)} = r, \tag{2.22}$$

and q and r are bounded where $k^{1/3} | x - \tilde{x} | \geqq B$ by hypothesis. We shall estimate the derivatives of q and r with respect to ψ. From (2.21) and

(2.22),

$$\frac{dq}{d\psi} = -q^2 r, \tag{2.23}$$

$$\frac{dr}{d\psi} = -r^2 q + q^2 \frac{\varphi'''(x)}{\sigma}. \tag{2.24}$$

By repeated differentiation of (2.23) and (2.24), we see that all derivatives of q and r with respect to ψ are bounded. Now we introduce ψ as variable of integration, to obtain

$$I_2 = \frac{e^{ik\varphi(\tilde{x})}}{k^{1/3}} \int e^{i\sigma\psi}[1 - p(\psi)]g(x(\psi))\, q d\psi. \tag{2.25}$$

Integrating by parts n times, we obtain

$$I_2 = \frac{i^n e^{ik\varphi(\tilde{x})}}{k^{1/3}\sigma^n} \int e^{i\sigma\psi} \left(\frac{\partial}{\partial\psi}\right)^n [(1 - p(\psi))g(x(\psi))q]\, d\psi. \tag{2.26}$$

Since all derivatives which appear under the integral sign are bounded, the theorem is proved.

We remark that only $2N + 2$ derivatives of g and $2N + 4$ derivatives of φ are required for Theorem 1 to be valid. However, such weaker hypotheses would make the estimation of I_2 more tedious.

LEMMA. *If $g(x, \lambda)$ and $\varphi(x, \lambda)$ are infinitely differentiable with respect to x and all derivatives of φ and g are continuous with respect to λ, g has compact support in x and λ, $\varphi'(\tilde{x}, \lambda) = 0$ and $B > 0$, then there exists a function $B_1(\lambda)$ such that q and r are bounded independently of k and λ if $k^{-1/3}B < |x - \tilde{x}| \leqq B_1(\lambda)$.*

Proof. We choose $B_1(\lambda)$ such that

$$\left|\frac{\varphi'(x, \lambda)}{(x - \tilde{x})\varphi''(\tilde{x}, \lambda)}\right| \geqq \frac{1}{2} \quad \text{and} \quad \left|\frac{\varphi''(x, \lambda)}{\varphi''(\tilde{x}, \lambda)}\right| < 2, \tag{2.27}$$

where $|x - \tilde{x}| < B_1(\lambda)$. We have the identity

$$q = \frac{\sigma}{k^{2/3}\varphi'(x, \lambda)} = \frac{1}{2k^{1/3}(x - \tilde{x}) \dfrac{\varphi'(x, \lambda)}{(x - \tilde{x})\varphi''(\tilde{x}, \lambda)}}, \tag{2.28}$$

and thus,

$$|q| \leqq \frac{1}{k^{1/3}|x - \tilde{x}|} \leqq \frac{1}{B} \quad \text{if} \quad k^{-1/3}B \leqq |x - \tilde{x}| \leqq B_1(\lambda). \tag{2.29}$$

Similarly,

$$r = \frac{\varphi''(x, \lambda)}{k^{1/3}\varphi'(x, \lambda)} = \frac{\varphi''(x, \lambda)/\varphi''(\tilde{x}, \lambda)}{k^{1/3}(x - \tilde{x}) \dfrac{\varphi'(x, \lambda)}{(x - \tilde{x})\varphi''(\tilde{x}, \lambda)}}, \tag{2.30}$$

and thus,

$$(2.31) \qquad |r| \leqq \frac{4}{k^{1/3}|x-\tilde{x}|} \leqq \frac{4}{B} \quad \text{if} \quad k^{-1/3}B \leqq |x-\tilde{x}| \leqq B_1(\lambda).$$

3. An endpoint. This section is devoted to the proof of the following theorem.

THEOREM 2. *If $\varphi(x, \lambda)$ and $g(x, \lambda)$ are infinitely differentiable with respect to x and all derivatives of φ and g are continuous with respect to λ, g has compact support in x and λ, and if $\varphi'(0, \lambda)/\varphi'(x, \lambda)$ is bounded independently of λ in the support of g, then, for any N,*

$$(3.1) \qquad \begin{aligned} I(k,\lambda) &= \int_0^\infty e^{ik\varphi(x,\lambda)}x^r g(x,\lambda)\,dx \\ &= \frac{e^{ik\varphi(0,\lambda)}}{k^{(1+r)/2}\tau^{1+r}}\left[\sum_{j=0}^{N}\frac{d_j(\lambda)}{\tau^{2j+1}} + O\left(\frac{1}{\tau^{2N+2}}\right)\right], \end{aligned}$$

where r is a fixed complex number, $\operatorname{Re} r > -1$ *(to ensure existence of the integral), τ is defined by*

$$(3.2) \qquad \tau = k^{1/2}|\varphi'(0,\lambda)|$$

and the coefficients d_j involve g and its derivatives up to order j at $x = 0$ and derivatives of φ up to order $j+1$ at $x = 0$.

The proof of Theorem 2 is analogous to the proof of Theorem 1. We shall not indicate the presence of the parameter λ explicitly. We define ψ and y by means of

$$(3.3) \qquad \tau\psi = k|\varphi(x) - \varphi(0)|,$$

$$(3.4) \qquad y = k^{1/2}x.$$

Taylor's theorem implies that

$$(3.5) \qquad k|\varphi(x) - \varphi(0)| = \tau\psi = \tau y + y^2 f\left(y\frac{\varphi'(0)}{\tau}\right)$$

with an appropriate f, and thus,

$$(3.6) \qquad \frac{\psi}{\tau} = \frac{y}{\tau} + \frac{y^2}{\tau^2}f\left(y\frac{\varphi'(0)}{\tau}\right).$$

If y is bounded and τ is large enough, we can apply the implicit function theorem to obtain

$$(3.7) \qquad \frac{y}{\tau} = \frac{\psi}{\tau}\left(1 + \frac{\psi}{\tau}f_1\left(\frac{\psi}{\tau}\right)\right),$$

$$\frac{dy}{d\psi} = 1 + \frac{\psi}{\tau} f_2\left(\frac{\psi}{\tau}\right) \tag{3.8}$$

with appropriate f_1 and f_2.

Now we choose $p \in C^\infty$ such that

$$p(\psi) = \begin{cases} 1 & \text{if} \quad \psi \leqq 1, \\ 0 & \text{if} \quad \psi > 2. \end{cases} \tag{3.9}$$

Then (3.1) can be rewritten as

$$\begin{aligned} I(k) &= \int_0^\infty e^{ik\varphi} p(\psi) x^r g(x)\, dx \\ &\qquad + \int_0^\infty e^{ik\varphi}(1 - p(\psi)) x^r g(x)\, dx = I_1 + I_2. \end{aligned} \tag{3.10}$$

Introducing ψ as a variable of integration in I_1, we have

$$I_1 = \frac{e^{ik\varphi(0)}}{k^{1/2}} \int_0^\infty e^{\pm i\tau\psi} p(\psi)(yk^{-1/2})^r g(k^{-1/2}y) \frac{dy}{d\psi}\, d\psi, \tag{3.11}$$

where $\pm = \operatorname{sgn}(\varphi(x) - \varphi(0))$ and $dy/d\psi$ is given by (3.8). After expanding $y^r g(dy/d\psi)$ in powers of ψ/τ, we obtain

$$\begin{aligned} I_1 &= \frac{e^{ik\varphi(0)}}{k^{(1+r)/2}\tau^{r+1}} \\ &\cdot \left[\sum_{j=0}^N \int_0^\infty e^{\pm i\tau\psi} p(\psi) c_j \frac{\psi^{j+r}}{\tau^j}\, d\psi + \int_0^\infty e^{\pm i\tau\psi} p(\psi) \frac{\psi^{r+N+1}}{\tau^{N+1}} h\left(\frac{\psi}{\tau}\right) dx \right]. \end{aligned} \tag{3.12}$$

The integrals which appear in (3.12) may be treated by the usual endpoint procedure (see Erdélyi [2]), and we obtain

$$I_1 = \frac{e^{ik\varphi(0)}}{k^{(1+r)/2}\tau^{r+1}} \left[\sum_{j=0}^N \frac{d_j}{\tau^{2j}} + O\left(\frac{1}{\tau^{2N+2}}\right) \right], \tag{3.13}$$

where

$$d_j = \Gamma(j + r + 1) e^{\pm i(j+r+1)\pi/2} c_j. \tag{3.14}$$

The proof of Theorem 2 will be completed if we estimate I_2. We define

$$q = \frac{\tau}{k^{1/2} \mid \varphi'(x) \mid} = \left| \frac{\varphi'(0)}{\varphi'(x)} \right|; \tag{3.15}$$

by hypothesis, q is bounded independently of λ in the support of g. We have

$$\frac{dq}{d\psi} = \mp q^3 \frac{\varphi''(x)}{\tau}, \tag{3.16}$$

$$\frac{dx}{d\psi} = \frac{q}{k^{1/2}} = q\,\frac{|\varphi'(0)|}{\tau}, \tag{3.17}$$

and hence,

$$\frac{d^n q}{d\psi^n} = O\left(\frac{1}{\tau^n}\right). \tag{3.18}$$

Introducing ψ as variable of integration in I_2, we obtain

$$I_2 = \frac{e^{ik\varphi(0)}}{k^{1/2}} \int_0^\infty e^{\pm i\tau\psi}[1 - p(\psi)]g(x(\psi))x^r q\, d\psi. \tag{3.19}$$

Using (3.4) and (3.6), x^r can be expressed in terms of $k^{-r/2}\psi^r f(\psi)$ with an appropriate f. After integrating by parts n times, we obtain

$$I_2 = \frac{e^{ik\varphi(0)}}{k^{(r+1)/2}} \frac{i^n}{\tau^n} \int_0^\infty e^{\pm i\tau\psi} \left(\frac{d}{d\psi}\right)^n [(1 - p(\psi))\psi^r gq]\, d\psi, \tag{3.20}$$

and the theorem is proved.

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