ON THE STATIONARY PHASE APPROXIMATION OF CHIRP SPECTRA

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ABSTRACT

The use of the stationary phase principle is often advocated for evaluating the spectrum of a chirp. This issue is considered here in detail, especially with respect to the quantitative control of the corresponding approximation error. A careful analysis leads to the introduction of a refined criterion, which turns out to be much more complicated than the heuristic conditions which are usually considered in this context. It is moreover evidenced, by means of two counterexamples belonging to the important class of power-law chirps, that — as opposed to a common belief — usual heuristic conditions are by themselves neither necessary nor sufficient for validating a stationary phase approximation.

1. MOTIVATION

Monocomponent AM/FM signals are ubiquitous in Nature (animal communication, geophysics, astrophysics, acoustics,...) as well as in man-made systems (radar, sonar, seismic exploration,...) : "chirps" have been introduced for serving as natural models to such waveforms.

Definition 1 *By definition, chirps are signals of the form*

$$x(t) = a(t)\,\exp\{i\varphi(t)\}, \tag{1}$$

where $a(t)$ is some positive, low-pass and smooth amplitude function whose evolution is slow as compared to the oscillations of the phase $\varphi(t)$.

More precisely, conditions on $a(t)$ and $\varphi(t)$ usually read (see, e.g., [2, 3, 6, 9])

$$\varepsilon_1 = \left|\frac{\dot{a}(t)}{a(t)\,\dot{\varphi}(t)}\right| \ll 1\,;\;\; \varepsilon_2 = \left|\frac{\ddot{\varphi}(t)}{\dot{\varphi}^2(t)}\right| \ll 1, \tag{2}$$

where the first condition guarantees that, over a (local) pseudo-period $T(t) = 2\pi/\dot{\varphi}(t)$, the amplitude $a(t)$ experiences almost no relative change, whereas the second condition imposes that $T(t)$ itself is slowly-varying, thus giving sense to the notion of a pseudo-period.

Although the definition of a chirp is usually given in the time domain (as in (1)), many applications call for a companion description in the frequency domain [1, 3]. In this respect, it is customary to make use of a stationary phase approximation, assuming more or less explicitly that the conditions given in (2) support the effectiveness of the approach.

2. STATIONARY PHASE APPROXIMATION OF CHIRP SPECTRA

2.1. The stationary phase principle

The argument of the stationary phase principle can be phrased as follows. Let I be an oscillatory integral of the form

$$I = \int_\Omega b(t)\,e^{i\psi(t)}\,dt, \tag{3}$$

where both $b(t) > 0$ and $\psi(t)$ are C^1, whereas $\mathrm{supp}\{\psi(t)\}$ is restricted to some interval $\Omega \subset \mathbb{R}$ over which $b(t)$ is integrable. Assuming that $b(t)$ is slowly-varying as compared to the oscillations controlled by $\psi(t)$, positive and negative values of the integrand tend to cancel each other, with the consequence that the main contribution to I only comes from the vicinity of those points where the derivative of the phase is zero.

2.2. One stationary point and approximated chirp spectra

In the case of the model (3), classical results from stationary phase theory (see, e.g. [7]) cannot be applied directly since oscillations are not controlled by some natural parameter of large value. Assuming however that the phase $\psi(t)$ has one and only one non-degenerate stationary point t_s (i.e., that we have $\dot{\psi}(t_s) = 0$ and $\ddot{\psi}(t_s) \neq 0$), we can make the change of variables

$$u^2 = \frac{\psi(t) - \psi(t_s)}{\ddot{\psi}(t_s)/2}$$

so as to rewrite (3) in the canonical form

$$I = e^{i\psi(t_s)} \int_{\Omega'} g(u)\, e^{i\beta u^2}\, du, \qquad (4)$$

with $g(u) = b(t(u))(du/dt)^{-1}$ and $\beta = \ddot{\psi}(t_s)/2$. Using a Taylor expansion for the exponential in the right-hand side of (4), we are thus led [4] to decomposing (3) as $I = I_a + R$, with

$$I_a = \sqrt{\frac{2\pi}{|\ddot{\psi}(t_s)|}}\, b(t_s)\, e^{i\psi(t_s)}\, e^{i(\mathrm{sgn}\ddot{\psi}(t_s))\pi/4}, \qquad (5)$$

the quality of using I_a as an approximation for I depending on the magnitude of the remainder R.

Extending an approach developed in [4, 8] allows for bounding explicitly the relative error $Q = |R/I_a|$ as

$$Q \leq Q_m = \frac{5}{4} \frac{\sup_{u \in \Omega'} |\ddot{g}|}{|\beta|\, g(t_s)} \qquad (6)$$

and the stationary phase approximation is therefore valid if $Q_m \ll 1$. Given the model (3), an explicit evaluation of this quantity leads to $Q_m = \sup_{t \in \Omega} F(t)$, where $F(t)$ is a fairly complicated function which is explicitly given in [1] and which depends non-linearly on $b(t)$, $\psi(t)$ and some of their derivatives up to third order.

This result provides us with a sufficient criterion for (quantitatively) justifying the effectiveness of the approximation and it can be readily applied to the problem of evaluating the spectrum of a chirp (1) with a monotonic instantaneous frequency by setting $b(t) = a(t)$ and $\psi(t) = \varphi(t) - 2\pi f t$ (with the stationary point t_s thus defined by $\dot{\varphi}(t_s) = 2\pi f$). What turns out is that the corresponding error is not only controlled by the terms ε_1 and ε_2 (as defined in (2)), but also by additional terms depending on more complicated combinations of $a(t)$, $\varphi(t)$ and some of their higher-order derivatives. In general, evaluating the upper bound Q_m in (6) appears not to be feasible but, in most cases, a useful substitute is given by $F(t_s)$, such a simplification corresponding in fact to only considering the leading term in the integral remainder R.

2.3. No stationary point and quasi-analyticity of chirps

Returning to the general model (3), it is finally worth investigating the case where there is no stationary point. In such a situation where $\dot{\psi}(t) \neq 0$ for any t, (3) can be rewritten as

$$I = \int_{\Omega} \frac{b(t)}{i\dot{\psi}(t)}\, i\dot{\psi}(t)\, e^{i\psi(t)}\, dt,$$

so as to be integrated by parts. Assuming that $b(t) \in L^1(\Omega)$ and $b(\partial\Omega) = 0$, we get that

$$\frac{I}{\|b\|_1} \leq \left\| \frac{\dot{b}(t)}{b(t)\dot{\psi}(t)} \right\|_\infty + \left\| \frac{\ddot{\psi}(t)}{\dot{\psi}^2(t)} \right\|_\infty, \qquad (7)$$

which means that, as compared to the situation where the oscillations of the phase would be infinitely slowed down, the magnitude of (3) is in this case bounded from above by a quantity whose decay to zero is controlled by chirp-like conditions. Moreover, in the case where I corresponds to the Fourier transform of the chirp (1), i.e. when $b(t) = a(t)$ and $\psi(t) = \varphi(t) - 2\pi f t$, and if we furthermore assume that $\dot{\varphi}(t) > 0$ for any $t \in \Omega$, we can conclude that the frequency domain for which no stationary point exists is the half-line of negative frequencies. Since we have in this case $\ddot{\psi}(t) = \ddot{\varphi}(t)$ and $\dot{\psi}(t) \geq \dot{\varphi}(t)$ when $f < 0$, we are ensured that

$$\left\| \frac{\dot{b}(t)}{b(t)\dot{\psi}(t)} \right\|_\infty \leq \left\| \frac{\dot{a}(t)}{a(t)\dot{\varphi}(t)} \right\|_\infty$$

and

$$\left\| \frac{\ddot{\psi}(t)}{\dot{\psi}^2(t)} \right\|_\infty \leq \left\| \frac{\ddot{\varphi}(t)}{\dot{\varphi}^2(t)} \right\|_\infty.$$

It appears therefore that the heuristic conditions (2) are sufficient for making the right-hand side of (7) negligible, thus guaranteeing the quasi-analyticity of the exponential model (1) — in the sense that spectral contributions at negative frequencies are almost zero —, with the consequence that the quantity $\dot{\varphi}(t)/2\pi$ can be effectively interpreted as the instantaneous frequency of the chirp.

3. EXAMPLES AND COUNTER-EXAMPLES

3.1. Power-law chirps

In order to evidence the possible limitations in using a stationary phase approximation when evaluating a chirp spectrum, we focus on the important class of "power-law" chirps [1, 3] :

Definition 2 *By definition, a power-law chirp is a chirp (1), in which*

$$a(t) = (t_0 - t)^{-\alpha}$$

and

$$\varphi(t) = 2\pi d (t_0 - t)^{\beta},$$

with α, β and d real-valued parameters and $t < t_0$.

From this definition, it has to be remarked that different types of waveforms can be obtained, depending on the values of the parameters α and β :

- considering $a(t)$ as the amplitude of the chirp, we will observe that $a(t_0) = 0$ (resp. $+\infty$) if $\alpha < 0$ (resp. > 0) ;
- identifying $\dot{\varphi}(t)/2\pi = d\beta(t_0 - t)^{\beta - 1}$ with the "instantaneous frequency" of the chirp leads to a power-law divergence in t_0 for all β's such that $\beta < 1$. This

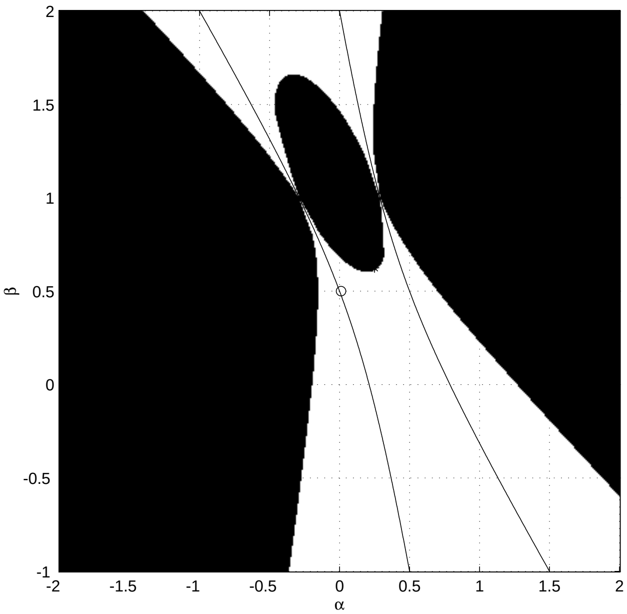

Figure 1: Comparison of heuristic and refined stationary phase criteria for power-law chirps of indices α and β (see text) — The white (resp. gray) domain corresponds to values of α and β such that the refined criterion C_2 is smaller (resp. larger) than the heuristic one C_1. Full lines correspond to the exact condition $C_2 = 0$. The cross, the circle and the star are the specific values used in Figures 2 to 4, respectively.

> will however correspond to an "indefinitely oscillating" signal in t_0 only if we have the stronger condition $\beta < 0$ [5]. In fact, within the range $0 < \beta < 1$, the phase does present a well-defined value in t_0 : $\varphi(t_0) = 0$, thus connecting the singular behaviour of its derivative with a non-oscillating singularity of the waveform in t_0.

In the case of power-law chirps, it can be shown that both criteria, derived from either the heuristic conditions (2) or the refined analysis sketched in Section 2, share the same frequency dependence

$$\varepsilon = C\,(\beta d)^{\frac{1}{\beta-1}}\, f^{-\frac{\beta}{\beta-1}}, \tag{8}$$

with the only difference stemming from the pre-factor C, which reads

$$C_1 = (1/2\pi)\max(|\alpha|, |\beta - 1|)$$

in the first case and

$$C_2 = (5/48\pi)|12\alpha^2 - 12\alpha + 12\alpha\beta + 2\beta^2 - 5\beta + 2|/|\beta - 1|$$

in the second one.

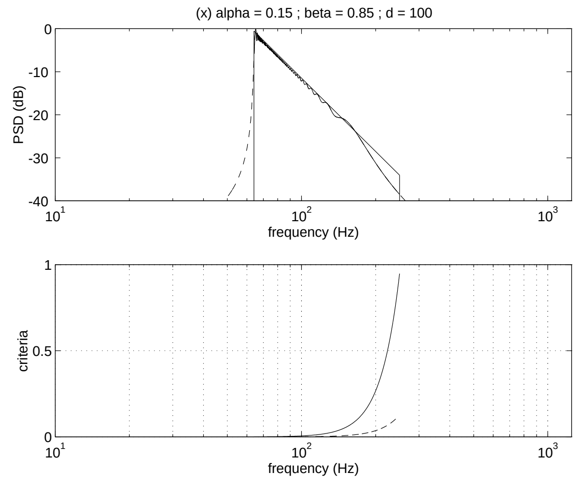

Figure 2: Validity of the stationary phase approximation — Counter-example 1 : in the case of a power-law chirp with parameters identified by the cross in Figure 1, we observe that the heuristic criterion (dotted line in the bottom diagram) predicts a good approximation (in the chosen frequency band), whereas the comparison with the actual spectrum (top diagram) reveals a significative difference, as to be expected from the refined criterion (full line in the bottom diagram).

Depending on which of these quantities is greater, we can therefore evidence, for any given d, pairs (α, β) such that the stationary phase approximation still remains valid whereas the heuristic conditions (2) are violated or, on the contrary, such that the approximation breaks down whereas the same conditions are satisfied. This is illustrated in Figures 1 to 3.

3.2. Gravitational waves chirps

Finally, we can remark that, in the important context of gravitational waves data analysis [1, 3], power-law chirps are extensively used for the modelling of radiations expected to be observed from the coalescence of very massive binary objects (typically, neutron stars). The detection of the corresponding waveforms is a challenging problem for which direct implementations of matched filtering techniques in the time domain may prove prohibitive from the point of view of the computational load. In this respect, alternative approaches have been proposed, whose reduced complexity is based on FFT implementations, thus calling for an accurate frequency description of the expected waveform [3, 1].

In the considered astrophysical application, the indices of the chirps have to be fixed — based on physical arguments — to $\alpha = 1/4$ (blow up of the amplitude) and $\beta = 5/8$ (non-oscillating singularity), whereas the chirp rate d is a free parameter which is related to the masses of the sys-

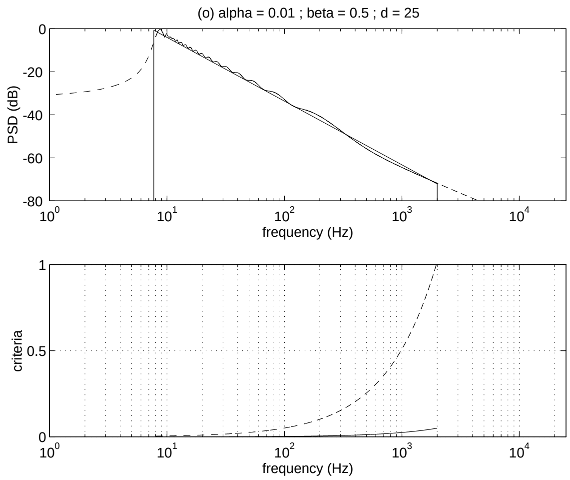

Figure 3: Validity of the stationary phase approximation — Counter-example 2 : in the case of a power-law chirp with parameters identified by the circle in Figure 1, we observe that the heuristic criterion (dotted line in the bottom diagram) predicts a poor approximation (in the chosen frequency band), whereas the comparison with the actual spectrum (top diagram) reveals an excellent agreement, as to be expected from the refined criterion (full line in the bottom diagram).

tem. It turns out that, with the right values for α and β, and over a wide range of meaningful values for d, criteria C_1 and C_2 almost coincide. Moreover, they both have a small value, thus supporting a posteriori the effectiveness of the stationary phase approximation which is commonly used in this context (see e.g. [3]). This is illustrated in Figure 4.

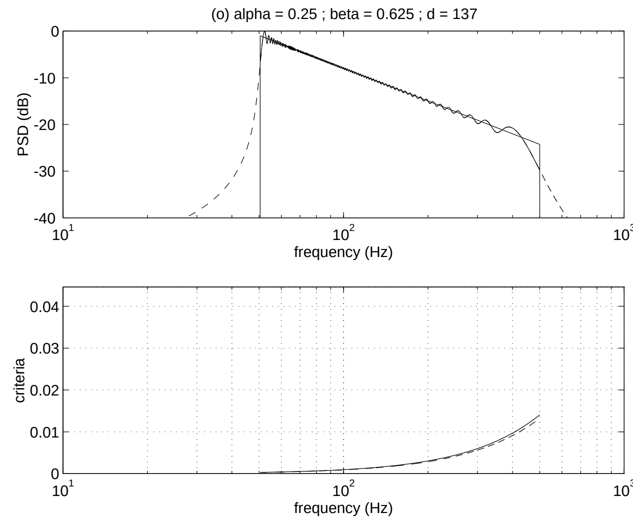

Figure 4: Validity of the stationary phase approximation — Example of a typical gravitational wave model : in the case of a power-law chirp with parameters fixed to $\alpha = 1/4$, $\beta = 5/8$ and $d = 137$, we observe that the heuristic and refined criteria almost coincide (bottom diagram). Moreover, they both have a small value in the considered frequency band, thus supporting a posteriori the stationary phase approximation which is commonly used in this context.

4. REFERENCES

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