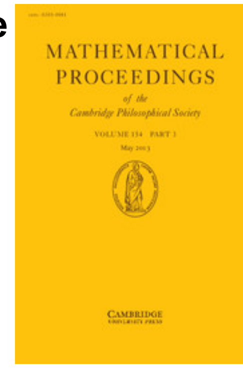

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AN EXTENSION OF THE METHOD OF STEEPEST DESCENTS

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ABSTRACT. In the integral

$$\int g(z)\exp\{Nf(z,\alpha)\}\,dz$$

the functions $g(z), f(z,\alpha)$ are analytic functions of their arguments, and N is a large positive parameter. When N tends to ∞, asymptotic expansions can usually be found by the method of steepest descents, which shows that the principal contributions arise from the saddle points, i.e. the values of z at which $\partial f/\partial z = 0$. The position of the saddle points varies with α, and if for some α (say $\alpha = 0$) two saddle points coincide (say at $z = 0$) the ordinary method of steepest descents gives expansions which are not uniformly valid for small α. In the present paper we consider this case of two nearly coincident saddle points and construct uniform expansions as follows. A new complex variable u is introduced by the implicit relation

$$f(z,\alpha) = \tfrac{1}{3}u^3 - \zeta(\alpha)\,u + A(\alpha),$$

where the parameters $\zeta(\alpha)$, $A(\alpha)$ are determined explicitly from the condition that the (u, z) transformation is uniformly regular near $z = 0, \alpha = 0$ (see § 2 below). We show that with these values of the parameters there is one branch of the transformation which is uniformly regular. By taking u on this branch as a new variable of integration we obtain for the integral uniformly asymptotic expansions of the form

$$\exp\{-NA(\alpha)\}\left\{\frac{\mathrm{Ai}(N^{\frac{2}{3}}\zeta)}{N^{\frac{1}{3}}}\sum_{s=0}^{\infty}\frac{a_s(\alpha)}{N^s} + \frac{\mathrm{Ai}'(N^{\frac{2}{3}}\zeta)}{N^{\frac{2}{3}}}\sum_{s=0}^{\infty}\frac{b_s(\alpha)}{N^s}\right\},$$

where Ai and Ai′ are the Airy function and its derivative respectively, and $A(\alpha)$, $\zeta(\alpha)$ are the parameters in the transformation. The application to Bessel functions of large order is briefly described.

1. *Introduction. The method of steepest descents.* The present paper is concerned with the problem of finding the asymptotic expansion of contour integrals of the form

$$\int g(z)\exp\{Nf(z)\}\,dz, \tag{1·1}$$

where N is a large real positive parameter, and $f(z)$ and $g(z)$ are analytic functions of z. The asymptotic expansion can often be found by the *method of steepest descents*, as follows. The contour is deformed to pass through *saddle points* of $f(z)$, which are the zeros of the derivatives $f'(z)$. For the sake of simplicity let us suppose that all zeros of $f'(z)$ are simple; zeros of higher order introduce no serious complication. Let z_0 denote a typical saddle point, and for a contour through z_0 let a new local variable u be introduced by the equation

$$-\tfrac{1}{2}u^2 = f(z) - f(z_0) = \tfrac{1}{2}(z-z_0)^2 f''(z_0) + \ldots,$$

whence

$$f'(z)\frac{dz}{du} = -u.$$

Since $f''(z_0) \neq 0$, each branch of the transformation is regular and (1, 1) in a domain about $z = z_0$ (see (2), Theorem 115). There is therefore an expansion of the form

$$g(z)\frac{dz}{du} = -g(z)\frac{u(z)}{f'(z)} = \Sigma c_m u^m, \tag{1·2}$$

valid in small circles, $|z-z_0| \leqslant R_z$ and $|u| \leqslant R_u$, say. Because $f'(z)$ occurs in the denominator of (1·2), the radius of convergence in the z-plane is in general not greater than the distance from z_0 to the nearest saddle point. The contour of integration in the u-plane is now chosen to be the real axis near $u = 0$, and this choice corresponds to the curve of steepest descent in the z-plane. If $0 < U < R_u$, the segment $(-U, U)$ corresponds to an arc $C(U)$ in the z-plane passing through z_0, and

$$\exp\{-Nf(z_0)\}\int_{C(U)} g(z)\exp\{Nf(z)\}\,dz = \int_{-U}^{U} g(z)\frac{dz}{du}\exp(-\tfrac{1}{2}Nu^2)\,du$$

$$= \int_{-U}^{U}(\Sigma c_m u^m)\exp(-\tfrac{1}{2}Nu^2)\,du = \Sigma c_m N^{-\frac{1}{2}(m+1)}\int_{-N^{\frac{1}{2}}U}^{N^{\frac{1}{2}}U} v^m \exp(-\tfrac{1}{2}v^2)\,dv,$$

a convergent expansion in terms of incomplete factorial functions. If the limits of integration are formally replaced by $\pm\infty$ (since N is large) the expansion

$$\int g(z)\exp\{Nf(z)\}\,dz \sim \exp\{Nf(z_0)\}\,\Sigma c_m N^{-\frac{1}{2}(m+1)}\int_{-\infty}^{\infty} v^m\exp(-\tfrac{1}{2}v^2)\,dv \qquad (1\cdot3)$$

is obtained, which is usually not convergent. Usually only the neighbourhood of saddle points contributes significantly (see Jeffreys and Jeffreys (2), §17·04; Doetsch (1)). The complete asymptotic expansion is obtained by adding the contributions (1·3) from all relevant saddle points; this is the method of steepest descents. For extensions and details see Doetsch (1).

Suppose next that the function $f(z) = f(z, \alpha)$ contains a parameter α, and that α is allowed to vary in a domain of the complex α-plane. Then the saddle points vary with α, and it may happen that for some value of α (say $\alpha = 0$) two saddle points coincide, say at $z = 0$. If α is kept fixed, either at $\alpha = 0$ or at another value, the method of steepest descents is applicable for sufficiently large N (depending on α). But if we want expansions uniformly valid in a domain containing $\alpha = 0$, these cannot be found by steepest descents since the radius of convergence of (1·2) tends to zero with α. The present paper is concerned with the extension of the method of steepest descents to two nearly coincident saddle points.

Many examples are already known of expansions which are uniformly valid near $\alpha = 0$. Such expansions can be found for any solution of certain general types of ordinary linear second-order differential equations, and some of these (e.g. Bessel functions) can also be expressed as integrals. For an elegant version of this theory, which is completely rigorous, and for an account of earlier work see Olver (5). The expansions are in terms of Airy functions and their first derivatives, the typical Airy function being

$$\mathrm{Ai}(\zeta) = \frac{1}{2\pi i}\int_{\infty\exp(-\frac{1}{3}\pi i)}^{\infty\exp(\frac{1}{3}\pi i)} \exp(\tfrac{1}{3}u^3 - \zeta u)\,du, \qquad (1\cdot4)$$

where the integrand is the exponential function of a cubic polynomial. For a short account of Airy functions see Olver (5), (6); Jeffreys and Jeffreys (2), §17·07; Miller (4).

2. *The method of cubic representation.* In the present paper similar results are obtained directly from the integral. The method is applicable whether the integral satisfies a differential equation or not; conversely, there are differential equations

which have no known solution of the form (1·1), and so both methods are needed. The method now to be described expresses $f(z)$ as a cubic in a new variable u, a transformation suggested by (1·4). The parameters are chosen to give a (1, 1) mapping $u \leftrightarrow z$ uniformly valid near $\alpha = 0$. Only the simplest case will be studied, that of *two* nearly coincident saddle points.

Let us try, then, to represent $f(z, \alpha)$ by the cubic

$$f(z, \alpha) = \tfrac{1}{3}u^3 - \zeta(\alpha)\, u + A(\alpha). \tag{2·1}$$

If this is to be a regular (1, 1) transformation we must have $dz/du \neq 0$ or ∞, where

$$f'(z, \alpha)\frac{dz}{du} = u^2 - \zeta(\alpha).$$

Now $f'(z, \alpha)$ vanishes at the two saddle points $z_1(\alpha), z_2(\alpha)$, while $u^2 - \zeta(\alpha)$ vanishes at $u = \pm\zeta^{\frac{1}{2}}(\alpha)$. If the transformation is to be regular, these points must correspond, and so we have from (2·1) the equations

$$f(z_1, \alpha) = -\tfrac{2}{3}\zeta^{\frac{3}{2}}(\alpha) + A(\alpha), \quad f(z_2, \alpha) = +\tfrac{2}{3}\zeta^{\frac{3}{2}}(\alpha) + A(\alpha),$$

to determine $\zeta(\alpha)$ and $A(\alpha)$. It will be shown that with these values the transformation $u \leftrightarrow z$ is indeed uniformly regular and (1, 1) near $u = 0$. Thus there is an expansion of the form

$$g(z)\frac{dz}{du} = \Sigma c_m(\alpha)\, u^m$$

uniformly valid near $u = 0$, and we have formally

$$\exp\{-NA(\alpha)\}\int g(z)\exp\{Nf(z, \alpha)\}\, dz \sim \int (\Sigma c_m(\alpha)\, u^m)\exp\{N(\tfrac{1}{3}u^3 - \zeta u)\}\, du$$

$$\sim \Sigma c_m(\alpha)\int u^m \exp\{N(\tfrac{1}{3}u^3 - \zeta u)\}\, du,$$

an expansion expressible in terms of Airy functions if the limits of integration are formally made to tend to infinity. Unfortunately, this cannot be regarded as an asymptotic expansion since successive terms do not tend to decrease as $N \to \infty$. So we write instead

$$g(z)\frac{dz}{du} = \Sigma p_m(\alpha)\,(u^2 - \zeta)^m + \Sigma q_m(\alpha)\, u(u^2 - \zeta)^m. \tag{2·2}$$

The coefficients in this expansion can be found by repeatedly differentiating and putting $z = z_1$, $u = \zeta^{\frac{1}{2}}$, and $z = z_2$, $u = -\zeta^{\frac{1}{2}}$. Then

$$\exp\{-NA(\alpha)\}\int g(z)\exp\{Nf(z, \alpha)\}\, dz$$

$$\sim \Sigma p_m \int (u^2 - \zeta)^m \exp\{N(\tfrac{1}{3}u^3 - \zeta u)\}\, du \tag{2·3}$$

$$+ \Sigma q_m \int u(u^2 - \zeta)^m \exp\{N(\tfrac{1}{3}u^3 - \zeta u)\}\, du. \tag{2·4}$$

This also is expressible in Airy functions, and here successive terms tend to decrease. Which Airy function is appropriate depends on the contour of integration, e.g. when the contour goes from $\infty e^{-\frac{1}{3}\pi i}$ to $\infty e^{\frac{1}{3}\pi i}$ the expressions (2·3) and (2·4) can be rearranged in the form

$$\frac{\mathrm{Ai}\,(N^{\frac{2}{3}}\zeta)}{N^{\frac{1}{3}}}\Sigma\frac{A_s(\zeta)}{N^{2s}}+\frac{\mathrm{Ai}'\,(N^{\frac{2}{3}}\zeta)}{N^{\frac{5}{3}}}\Sigma\frac{B_s(\zeta)}{N^{2s}}+\frac{\mathrm{Ai}'\,(N^{\frac{2}{3}}\zeta)}{N^{\frac{2}{3}}}\Sigma\frac{C_s(\zeta)}{N^{2s}}+\frac{\mathrm{Ai}\,(N^{\frac{2}{3}}\zeta)}{N^{\frac{4}{3}}}\Sigma\frac{D_s(\zeta)}{N^{2s}}, \qquad (2\cdot 5)$$

where the first two terms come from (2·3) and the other two from (2·4).

In the following sections of the paper these statements are discussed in greater detail. There are two principal results requiring proof. It must be shown that the implicit relation between u, z and α may be solved to give u as a regular function of z and α (Theorem 1 below). It must also be shown that the series (2·5) is asymptotic in the sense that the error in breaking off the series at any stage is of the same order as the first neglected term (Theorem 2). As an example the application to Bessel functions of large order will be briefly described.

3. *The regularity of the transformation.* The transformation is

$$T_\alpha:\quad f(z,\alpha)=\tfrac{1}{3}u^3-\zeta(\alpha)\,u+A(\alpha), \qquad (3\cdot 1)$$

where by hypothesis $f'(z)$ has two small zeros $z_1(\alpha)$, $z_2(\alpha)$ for small α. To each value of z there correspond three values of u. We have seen in § 2 that the transformation cannot be uniformly regular unless

$$A(\alpha)=\tfrac{1}{2}f(z_1)+\tfrac{1}{2}f(z_2),\quad \tfrac{2}{3}\zeta^{\frac{3}{2}}(\alpha)=\tfrac{1}{2}f(z_2)-\tfrac{1}{2}f(z_1),$$

and we shall see that with these values one branch of the transformation is indeed uniformly regular for small z and α; on this branch z_1 and z_2 correspond to $\zeta^{\frac{1}{2}}$ and $-\zeta^{\frac{1}{2}}$ respectively. The following case is typical. Suppose that near $z=0$ and $\alpha=0$ the function $f(z,\alpha)$ can be expanded in the form

$$f(z,\alpha)=a_0(\alpha)+a_1(\alpha)\,z+a_2(\alpha)\,z^2+a_3(\alpha)\,z^3+\dots,$$

where $a_0(\alpha)$, $a_1(\alpha),\dots$ are regular near $\alpha=0$. By hypothesis $a_1(0)=0$, $a_2(0)=0$, $a_3(0)\neq 0$. We suppose, without loss of generality, that $a_2(\alpha)\equiv 0$, $a_3(\alpha)\equiv\frac{1}{3}$, for these values can be achieved by a change of origin and of orientation of z. If $a_1(\alpha)$ has a simple zero at $\alpha=0$, we redefine α so as to make $a_1(\alpha)\equiv-\alpha$, and we then have the expansion

$$f(z,\alpha)=a_0(\alpha)-\alpha z+\tfrac{1}{3}z^3+a_4(\alpha)\,z^4+\dots. \qquad (3\cdot 2)$$

The saddle points are given by

$$0=\frac{\partial}{\partial z}f(z,\alpha)=-\alpha+z^2+4a_4(\alpha)\,z^3+\dots,$$

whence† $z_1(\alpha)=\alpha^{\frac{1}{2}}\mathfrak{p}_1(\alpha^{\frac{1}{2}}), z_2(\alpha)=-\alpha^{\frac{1}{2}}\mathfrak{p}_1(-\alpha^{\frac{1}{2}})$. On substituting these forms in (3·2) it is found that $A(\alpha)=\frac{1}{2}f(z_1)+\frac{1}{2}f(z_2)$ is a regular function of α, and that

$$\tfrac{4}{3}\zeta^{\frac{3}{2}}(\alpha)=f(z_2)-f(z_1)=\tfrac{4}{3}\alpha^{\frac{3}{2}}\mathfrak{p}_2(\alpha).$$

† The symbol $\mathfrak{p}(x)$ stands for a power series of the form $\mathfrak{p}(x)=1+\sum_{n=1}^{\infty}k_nx^n$, convergent for small x and with leading coefficient 1.

We choose the branch for which $\zeta(\alpha) = \alpha\mathfrak{p}_3(\alpha)$, a regular function of α vanishing at $\alpha = 0$. We now have to study the relation between u and z, and to this end we shall construct the Riemann surface of (3·1) near $z = 0$. We shall see that there are two singular points $Z_1(\alpha), Z_2(\alpha)$ near $z = 0$ and that at these there are square-root singularities. These points are not identical with $z_1(\alpha), z_2(\alpha)$. The Riemann surface, which consists of three sheets, has the same two sheets connected at $Z_1(\alpha)$ and $Z_2(\alpha)$, and the branch corresponding to the third sheet is therefore uniformly regular near $z = 0$ for all sufficiently small α. Formal proofs of these results will now be given.

LEMMA. *There are (small) radii R_z, R_α with the following properties:*

(i) *When $|\alpha| \leqslant R_\alpha$, each of the equations*

$$f(z,\alpha) = f(z_1,\alpha), \quad f(z,\alpha) = f(z_2,\alpha)$$

has exactly three roots inside $|z| = R_z$.

(ii) *When $|z|$ describes the circumference of the circle $|z| = R_z$, the transformation T_α (for each fixed α in $|\alpha| \leqslant R_\alpha$) defines three image curves, each of which is described once without double points.*

Proof. To prove the first part of the lemma, consider first the limiting case $\alpha = 0$ of the equation $f(z,\alpha) = f(z_1,\alpha)$. This is

$$0 = \tfrac{1}{3}z^3 + z^4\{a_4(0) + \ldots\}.$$

Choose $R_z^{(1)}$ so that $z^4\{a_4(0)+\ldots\} < \tfrac{1}{12}z^3$ in $|z| \leqslant R_z^{(1)}$. When $\alpha \neq 0$ the equation can be written

$$\tfrac{1}{3}z^3 = -\{a_4(0)\,z^4 + \ldots\} + \alpha z - \{(a_4(\alpha) - a_4(0))\,z^4 + \ldots\} - \alpha z_1 + \{a_4(\alpha)\,z_1^4 + \ldots\} + \tfrac{1}{3}z_1^3.$$

The modulus of the first term on the right-hand side is less than $\tfrac{1}{12}|z|^3$ on $|z| = R_z^{(1)}$; and, since $z_1 \to 0$ as $\alpha \to 0$, the moduli of the other terms are together less than $\tfrac{1}{12}|z|^3$ if α is small enough, say $|\alpha| \leqslant R_\alpha^{(1)}$. Rouché's theorem now shows that the equation has just as many roots inside $|z| \leqslant R_z^{(1)}$ as $\tfrac{1}{3}z^3$, i.e. there are three roots. A similar argument applies to the equation $f(z,\alpha) = f(z_2,\alpha)$, leading to radii $R_z^{(2)}$, $R_\alpha^{(2)}$. The first part of the lemma holds for any $R_z \leqslant \min(R_z^{(1)}, R_z^{(2)})$, and any $R_\alpha \leqslant \min(R_\alpha^{(1)}, R_\alpha^{(2)})$.

To prove the second part of the lemma, again consider first the limiting transformation T_0 corresponding to $\alpha = 0$. Then $\tfrac{1}{3}u^3 + A(0) = a_0(0) + \tfrac{1}{3}z^3 + a_4(0)z^4 + \ldots$, where $a_0(0) = A(0)$, whence $\tfrac{1}{3}u^3 = \tfrac{1}{3}z^3\{1 + 3a_4(0)\,z + \ldots\}$, and for small z we may take the cube root of both sides. There are three regular branches $u = z\mathfrak{p}_4(z)$, $u = e^{\frac{2}{3}\pi i}z\mathfrak{p}_4(z)$, $u = e^{\frac{4}{3}\pi i}z\mathfrak{p}_4(z)$. To fix ideas, consider the first branch. Since $|du/dz| = 1$ at $z = 0$, there is a radius $R_z < \tfrac{1}{2}\min(R_z^{(1)}, R_z^{(2)})$ such that this branch of the transformation is $(1,1)$ in $|z| \leqslant 2R_z$. The image of $|z| = R_z$ is a curve $C_0^{(1)}$ which does not pass through $u = 0$ and which is described just once without double points ((3), Theorem 115). (The same value of R_z can be used on the other two branches.) Take any point Z on $|z| = R_z$, and denote its image on the first branch by $U_0(Z)$, where $|U_0| > 0$. Then the image $U_\alpha(Z)$ of Z under the transformation T_α is uniquely determined for all sufficiently small α. For the equations defining U_0 and U_α are

$$f(Z,0) = \tfrac{1}{3}U_0^3 + A(0), \quad f(Z,\alpha) = \tfrac{1}{3}U_\alpha^3 - \zeta(\alpha)\,U_\alpha + A(\alpha),$$

and so

$$f(Z,\alpha) - f(Z,0) + A(0) - A(\alpha) + U_0\zeta(\alpha) = (U_0^2 - \zeta)(U_\alpha - U_0) + U_0(U_\alpha - U_0)^2 + \tfrac{1}{3}(U_\alpha - U_0)^3.$$

The left-hand side is small for small α, and the coefficient of $U_\alpha - U_0$ on the right-hand side does not vanish for small α. Thus ((3), Theorem 115, Corollary) there is a unique image $U_\alpha(Z)$ of any point Z on the circumference of the circle $|z| = R_z$, and clearly the dependence on α is uniformly regular for all small α and for all Z on the circumference. Denote the image curve by $C_\alpha^{(1)}$. For small α the slope, curvature, etc., of $C_\alpha^{(1)}$ clearly differ little from the slope, curvature, etc., of $C_0^{(1)}$, and the ratio of arc lengths $|dU_\alpha(Z)/dZ|$ is bounded away from zero. Thus for all sufficiently small α, say $|\alpha| \leqslant d^{(1)}$, the curve $C_\alpha^{(1)}$ is described just once without double points, and similarly the image curves on the other two sheets are described just once, if $|\alpha| \leqslant d^{(2)}$, $|\alpha| \leqslant d^{(3)}$ respectively. Take R_z as defined above, and take any $R_\alpha < \min(R_\alpha^{(1)}, R_\alpha^{(2)}; d^{(1)}, d^{(2)}, d^{(3)})$. These satisfy the conditions of the lemma.

THEOREM 1. *The transformation* (3·1) *has just one branch which is uniformly regular for small* z *and* α, *and on this branch the points* $z = z_1$, $z = z_2$ *correspond to* $u = \zeta^{\frac{1}{2}}$, $u = -\zeta^{\frac{1}{2}}$ *respectively. For small* α *the correspondence* $u \leftrightarrow z$ *is* (1, 1).

Proof. Consider the equation

$$\tfrac{1}{3}u^3 - \zeta(\alpha)\,u + A(\alpha) = f(z, \alpha).$$

For any fixed z and α this is a cubic equation for u, and the three values of u can be represented on a Riemann surface of three sheets. To find the branch points, the multiple points of the left-hand side must be found. They are at $\pm\zeta^{\frac{1}{2}}$. Thus the branch points are the roots of $f(z,\alpha) = -\frac{2}{3}\zeta^{\frac{3}{2}}(\alpha) + A(\alpha) = f(z_1, \alpha)$ and of $f(z,\alpha) = f(z_2, \alpha)$. For small α the lemma shows that there are exactly three roots of the first equation inside $|z| = R_z$, and clearly z_1 is a double root. Let the third root be denoted by Z_1. Similarly, the second equation has three roots z_2, z_2, Z_2. Thus the possible branch points inside $|z| = R_z$ are at z_1, Z_1; z_2, Z_2. We have already seen that $z_1 = \alpha^{\frac{1}{2}}\mathfrak{p}_1(\alpha^{\frac{1}{2}})$, $z_2 = -\alpha^{\frac{1}{2}}\mathfrak{p}_1(-\alpha^{\frac{1}{2}})$, and it is not difficult to show that $Z_1(\alpha) = -2\alpha^{\frac{1}{2}}\mathfrak{p}_5(\alpha^{\frac{1}{2}})$, $Z_2(\alpha) = 2\alpha^{\frac{1}{2}}\mathfrak{p}_5(-\alpha^{\frac{1}{2}})$.

Near z_1 we have

$$\tfrac{1}{3}u^3 - \zeta(\alpha)\,u = f(z,\alpha) - A(\alpha) = f(z_1,\alpha) - A(\alpha) + \tfrac{1}{2}(z-z_1)^2 f''(z_1,\alpha) + \dots,$$

whence
$$\tfrac{1}{3}(u-\zeta^{\frac{1}{2}})^2\,(u+2\zeta^{\frac{1}{2}}) = \tfrac{1}{2}(z-z_1)^2 f''(z_1,\alpha) + \dots,$$

where $f''(z_1,\alpha) = 2\alpha^{\frac{1}{2}}\mathfrak{p}_6(\alpha^{\frac{1}{2}}) \neq 0$, and so there are two solutions through $z = z_1$, $u = \zeta^{\frac{1}{2}}$ regular near $z = z_1$, and one solution through $z = z_1$, $u = -2\zeta^{\frac{1}{2}}$ regular near $z = z_1$. Thus $z = z_1$ is not a branch point of the Riemann surface.

Near Z_1 we have

$$\tfrac{1}{3}u^3 - \zeta(\alpha)\,u = f(z_1,\alpha) - A(\alpha) + (z-Z_1) f'(Z_1,\alpha) + \dots,$$

whence
$$\tfrac{1}{3}(u-\zeta^{\frac{1}{2}})^2\,(u+2\zeta^{\frac{1}{2}}) = (z-Z_1) f'(Z_1,\alpha) + \dots,$$

where $f'(Z_1,\alpha) \neq 0$, and so there are two solutions through $z = Z_1$, $u = \zeta^{\frac{1}{2}}$ with square-root branch points near $z = Z_1$, and one solution through $z = Z_1$, $u = -2\zeta^{\frac{1}{2}}$ regular near $z = Z_1$. Corresponding results hold for z_2, Z_2; we see that the only branch points are at Z_1, Z_2.

Let us now consider the arrangement of cuts on the Riemann surface. The second part of the lemma shows that no cut crosses $|z| = R_z$, and it is then clear that the only possible arrangement is the one where the square-root branches through Z_1 and Z_2 lie on the same two sheets. (This also follows because in the limit $\alpha = 0$ the Riemann surface degenerates into three separate sheets.) The branch on the third sheet is therefore uniformly regular in $|z| \leqslant R_z$. Apply the second part of the lemma to this branch. As z describes $|z| = R_z$, u on the regular sheet describes a curve once. Inside the circle u is a regular function of z (only on the regular sheet). Thus, by a known theorem ((3), Theorem 117), the relation $u \leftrightarrow z$ is (1, 1). Since we have already seen that on the regular branch $(z = Z_1) \leftrightarrow (u = -2\zeta^{\frac{1}{2}})$ we cannot have $(z = z_1) \leftrightarrow (u = -2\zeta^{\frac{1}{2}})$, and so we must have $(z = z_1) \leftrightarrow (u = \zeta^{\frac{1}{2}})$. Similarly, $(z = z_2) \leftrightarrow (u = -\zeta^{\frac{1}{2}})$ on the regular branch. On the regular branch there is an expansion in powers of z,

$$u = U_\alpha(Z) = \Sigma C_m(\alpha) z^m \quad \text{when} \quad |z| \leqslant R_z.$$

By Taylor's theorem, $$C_m(\alpha) = \frac{1}{2\pi i}\int_{|Z|=R_z} \frac{U_\alpha(Z)}{Z^{m+1}}\,dZ;$$

but $U_\alpha(Z)$ is a regular function of α on $|Z| = R_z$, and so $C_m(\alpha)$ is a regular function of α. Also, since the (z, u) relation is (1, 1), the series can be inverted uniformly. This concludes the proof of Theorem 1.

A more analytic proof of Theorem 1 may be preferred by some readers. If the equation

$$\tfrac{1}{3}u^3 - \zeta(\alpha)\,u + A(\alpha) = f(z, \alpha)$$

is transformed to the neighbourhood of the branch point by making the substitutions $z' = z - z_1$ and $u' = u - \zeta^{\frac{1}{2}}$, then, after a few trivial rotations and changes of scale it takes the form

$$u'^3 + 3au'^2 = bz'^2 + cz'^3\mathfrak{p}(z'), \tag{3·3}$$

where a, b, and c are functions of α such that, as α goes to zero, a and b go to zero, whereas c is bounded away from zero. Also, since the non-zero double point of the left-hand side, i.e. $u = -2a$ (henceforth we omit the primes on u and z), must correspond to the double point of the right-hand side (call it $z = \xi$), we must have the relation

$$4a^3 = b\xi^2 + c\xi^3\mathfrak{p}(\xi). \tag{3·4}$$

Since $z = \xi$ is a double point of the right-hand side it follows that

$$2b + 3c\xi\mathfrak{p}(\xi) + c\xi^2\mathfrak{p}'(\xi) = 0. \tag{3·5}$$

These two relations combined give

$$\frac{8a^3}{\xi^3} + c\mathfrak{p}(\xi) + c\xi\mathfrak{p}'(\xi) = 0. \tag{3·6}$$

Note that ξ goes to zero with α but that a/ξ is continuous and bounded away from zero for all values of α in some neighbourhood of zero, say for $|\alpha| < \alpha_0$.

Put

$$u = -2az\xi^{-1} + w$$

in (3·3) which with the help of (3·4) becomes

$$12a^2\xi^{-2}zw(z-\xi) + 3a\xi^{-1}w^2(\xi - 2z) + w^3 = 8a^3\xi^{-3}z^2(z-\xi) + cz^2\{z\mathfrak{p}(z) - \xi\mathfrak{p}(\xi)\}. \tag{3·7}$$

By Taylor's theorem

$$z\mathfrak{p}(z)-\xi\mathfrak{p}(\xi) = (z-\xi)\{\mathfrak{p}(\xi)+\xi\mathfrak{p}'(\xi)\}+(z-\xi)^2 P_2(z,\xi),$$

where $P_2(z,\xi)$ is an analytic function of z for $|\alpha| < \alpha_0$; therefore with the help of (3·6) the right-hand side of (3·7) becomes

$$cz^2(z-\xi)^2 P_2(z,\xi).$$

Put

$$w = z(\xi - z)\, v$$

in (3·7) which then reduces to

$$-12a^2\xi^{-2}v + 3a\xi^{-1}(\xi-2z)\,v^2 + z(\xi-z)\,v^3 = cP_2(z,\xi).$$

Since $a^2\xi^{-2}$ is bounded away from zero for $|\alpha| < \alpha_0$, while the coefficients of v^2 and v^3 tend to zero as z and α tend to zero, this equation defines v and consequently also u as a power series in z for $|z| < z_0$ say, with coefficients which are continuous in α for $|\alpha| < \alpha_0$. This concludes the alternative proof of Theorem 1.

Since $g(z)$ and dz/du are regular functions, there is also a uniform expansion (which was assumed in § 2)

$$g(z)\frac{dz}{du} = \Sigma p_m(\alpha)\,(u^2-\zeta)^m + \Sigma q_m(\alpha)\,u(u^2-\zeta)^m,$$

convergent for small u and α. The coefficients can be found by repeated differentiation and use of the correspondences $z_1 \leftrightarrow \zeta^{\frac{1}{2}}, z_2 \leftrightarrow -\zeta^{\frac{1}{2}}$. Now that the existence of these expansions has been established, the validity of the expansions can probably be extended by more detailed study of the transformation, but this aspect of the problem will not be pursued here.

4. *The functions* $F_m(\zeta, N, C_j), G_m(\zeta, N, C_j)$. These are defined by the equations

$$F_m(\zeta,N,C_j) = \frac{1}{2\pi i}\int_{C_j}(u^2-\zeta)^m \exp\{N(\tfrac{1}{3}u^3-\zeta u)\}\,du,$$

$$G_m(\zeta,N,C_j) = \frac{1}{2\pi i}\int_{C_j}u(u^2-\zeta)^m \exp\{N(\tfrac{1}{3}u^3-\zeta u)\}\,du.$$

The contours of integration are C_1 from $\infty\, e^{-\frac{1}{3}\pi i}$ to $\infty\, e^{\frac{1}{3}\pi i}$, or C_2 from $\infty\, e^{\frac{1}{3}\pi i}$ to $\infty\, e^{\pi i}$, or C_3 from $\infty\, e^{\pi i}$ to $\infty\, e^{-\frac{1}{3}\pi i}$. By the change of variable $N^{\frac{1}{3}}u = v$ we obtain

$$F_m(\zeta,N,C_j) = N^{-\frac{2}{3}m-\frac{1}{3}}F_m(N^{\frac{2}{3}}\zeta, 1, C_j),$$

$$G_m(\zeta,N,C_j) = N^{-\frac{2}{3}m-\frac{2}{3}}G_m(N^{\frac{2}{3}}\zeta, 1, C_j);$$

and by the change of variable $u = v\, e^{\frac{2}{3}\pi i}$ we obtain

$$F_m(\zeta,N,C_2) = \exp(\tfrac{4}{3}m\pi i+\tfrac{2}{3}\pi i)\,F_m(\zeta\, e^{\frac{4}{3}\pi i},N,C_1), \tag{4·1}$$

$$G_m(\zeta,N,C_2) = \exp(\tfrac{4}{3}m\pi i+\tfrac{4}{3}\pi i)\,G_m(\zeta\, e^{\frac{4}{3}\pi i},N,C_1). \tag{4·2}$$

The corresponding relations for C_3 are obtained by changing the sign of i. These equations show that we are actually dealing with a set of functions of the single variable $N^{\frac{2}{3}}\zeta$, and with integration along the single contour C_1, but it is convenient to keep the full notation. Clearly also

$$F_m(\zeta, N, C_1) + F_m(\zeta, N, C_2) + F_m(\zeta, N, C_3) = 0,$$

and there is a similar result for G_m. Straightforward integration by parts shows that

$$F_m(\zeta, N, C_j) = -\frac{2}{N}(m-1)\, G_{m-2}(\zeta, N, C_j),$$

$$G_m(\zeta, N, C_j) = -\frac{1}{N}\{(2m-1)\, F_{m-1} + 2(m-1)\,\zeta F_{m-2}\},$$

whence
$$F_m(\zeta, N, C_j) = \frac{2}{N^2}(m-1)\{(2m-5)\, F_{m-3} + 2(m-3)\,\zeta F_{m-4}\}, \tag{4·3}$$

$$G_m(\zeta, N, C_j) = \frac{2}{N^2}\{(2m-1)(m-2)\, G_{m-3} + 2(m-1)(m-3)\,\zeta G_{m-4}\}. \tag{4·4}$$

The factor $1/N^2$ outside the brackets shows that for bounded ζ the function F_m is of smaller order than F_{m-4} as $N \to \infty$, and that a similar result holds for G_m. Thus the sequence $\{F_m\}$ tends to decrease in sets of four (which suggests, however, that the functions are not quite the best possible). When F_0, F_1, G_0, G_1 are known, the higher functions can be found from the recurrence formulae; for example, for the contour C_1 the first few functions are

$$\begin{aligned}
F_0(\zeta, N, C_1) &= N^{-\frac{1}{3}}\,\mathrm{Ai}\,(N^{\frac{2}{3}}\zeta), & G_0(\zeta, N, C_1) &= -N^{-\frac{2}{3}}\,\mathrm{Ai}'\,(N^{\frac{2}{3}}\zeta),\\
F_1(\zeta, N, C_1) &= 0, & G_1(\zeta, N, C_1) &= -N^{-\frac{4}{3}}\,\mathrm{Ai}\,(N^{\frac{2}{3}}\zeta),\\
F_2(\zeta, N, C_1) &= 2N^{-\frac{5}{3}}\,\mathrm{Ai}'\,(N^{\frac{2}{3}}\zeta), & G_2(\zeta, N, C_1) &= -2\zeta N^{-\frac{4}{3}}\,\mathrm{Ai}\,(N^{\frac{2}{3}}\zeta),\\
F_3(\zeta, N, C_1) &= 4N^{-\frac{7}{3}}\,\mathrm{Ai}\,(N^{\frac{2}{3}}\zeta), & G_3(\zeta, N, C_1) &= -10N^{-\frac{8}{3}}\,\mathrm{Ai}'\,(N^{\frac{2}{3}}\zeta).
\end{aligned}$$

For the contours C_2, C_3 the functions Ai, Ai′ must be replaced by the appropriate Airy functions (see equation (4·1)). In the following section we shall be concerned not so much with the complete integrals F_m, G_m as with integrals between finite fixed limits independent of α and N. For large N these differ from the complete integrals by negligible exponentially small terms.

5. *Proof of the asymptotic expansion.* Suppose that the transformation $u \leftrightarrow z$ is regular and (1, 1) for $|\alpha| \leqslant R_\alpha$ in the closed circle $|u| \leqslant R_u$ which is assumed to contain the image of the circle $|z| \leqslant R_z$. The contribution to the integral (1·1) from the part of the contour outside $|z| = R_z$ is assumed to be negligibly small; this can usually be proved by the familiar arguments of the ordinary method of steepest descents. Then we have from equation (2·2)

$$\begin{aligned}
&\exp\{-NA(\alpha)\}\int g(z)\exp\{Nf(z,\alpha)\}\,dz\\
&= \sum_0^\infty p_m \int (u^2-\zeta)^m \exp\{N(\tfrac{1}{3}u^3 - \zeta u)\}\,du + \sum_0^\infty q_m \int u(u^2-\zeta)^m \exp\{N(\tfrac{1}{3}u^3-\zeta u)\}\,du,
\end{aligned} \tag{5·1}$$

where the integration with respect to u is over a finite arc lying inside $|u| = R_u$, and the integration with respect to z is over the image of the same arc. The two series on the right of (5·1) are absolutely and uniformly convergent. If each integral is formally replaced by the corresponding infinite integral, as in § 2, we obtain the series

$$2\pi i \Sigma p_m F_m(\zeta, N) + 2\pi i \Sigma q_m G_m(\zeta, N), \tag{5·2}$$

and each term can be expressed in terms of Ai and Ai′. To fix ideas, suppose that the contour in the u-plane is C_1; then (5·2) becomes after rearrangement

$$\frac{\mathrm{Ai}\,(N^{\frac{2}{3}}\zeta)}{N^{\frac{1}{3}}} \Sigma \frac{A_s(\zeta)}{N^{2s}} + \frac{\mathrm{Ai}'\,(N^{\frac{2}{3}}\zeta)}{N^{\frac{5}{3}}} \Sigma \frac{B_s(\zeta)}{N^{2s}} \tag{5·3}$$

$$+ \frac{\mathrm{Ai}'\,(N^{\frac{2}{3}}\zeta)}{N^{\frac{2}{3}}} \Sigma \frac{C_s(\zeta)}{N^{2s}} + \frac{\mathrm{Ai}\,(N^{\frac{2}{3}}\zeta)}{N^{\frac{4}{3}}} \Sigma \frac{D_s(\zeta)}{N^{2s}}, \tag{5·4}$$

where the terms (5·3) arise from the functions F_m and the terms (5·4) from the functions G_m. We wish to show that these formal series are asymptotic. Suppose then that each of the four series is cut off after the Mth term. We shall show that the error is of the same order as the first omitted term.

THEOREM 2. *When $N \to \infty$, then for fixed M and for all $|\alpha| \leqslant R_\alpha$*

$$\exp\{-NA(\alpha)\}\int g(z)\exp\{Nf(z,\alpha)\}\,dz$$

$$= \frac{\mathrm{Ai}\,(N^{\frac{2}{3}}\zeta)}{N^{\frac{1}{3}}}\left\{\sum_0^M \frac{A_s(\zeta)}{N^{2s}} + O\left(\frac{1}{N^{2M+2}}\right)\right\} + \frac{\mathrm{Ai}'\,(N^{\frac{2}{3}}\zeta)}{N^{\frac{5}{3}}}\left\{\sum_0^M \frac{B_s(\zeta)}{N^{2s}} + O\left(\frac{1}{N^{2M+2}}\right)\right\}$$

$$+ \frac{\mathrm{Ai}'\,(N^{\frac{2}{3}}\zeta)}{N^{\frac{2}{3}}}\left\{\sum_0^M \frac{C_s(\zeta)}{N^{2s}} + O\left(\frac{1}{N^{2M+2}}\right)\right\} + \frac{\mathrm{Ai}\,(N^{\frac{2}{3}}\zeta)}{N^{\frac{4}{3}}}\left\{\sum_0^M \frac{D_s(\zeta)}{N^{2s}} + O\left(\frac{1}{N^{2M+2}}\right)\right\}. \tag{5·5}$$

Proof. Consider the formal expansions (5·3) and (5·4) above. Equations (4·3) and (4·4) show that successive terms tend to decrease. The functions $A_0, \ldots, A_M$; $B_0, \ldots, B_M$; $C_0, \ldots, C_M$; $D_0, \ldots, D_M$ are therefore completely determined by a *finite* number of the coefficients $p_0, p_1, p_2, \ldots; q_0, q_1, q_2, \ldots$. Choose a finite number L so that these are included in the set p_s, q_s $(0 \leqslant s \leqslant L-1)$. L depends on M and will be defined more precisely later. Rewrite (5·1) in the form

$$\exp\{-NA(\alpha)\}\int g(z)\exp\{Nf(z,\alpha)\}\,dz$$

$$- \sum_0^{L-1} p_m \int (u^2-\zeta)^m \exp\{N(\tfrac{1}{3}u^3 - \zeta u)\}\,du - \sum_0^{L-1} q_m \int u(u^2-\zeta)^m \exp\{N(\tfrac{1}{3}u^3-\zeta u)\}\,du$$

$$= \sum_L^\infty p_m \int (u^2-\zeta)^m \exp\{N(\tfrac{1}{3}u^3-\zeta u)\}\,du + \sum_L^\infty q_m \int u(u^2-\zeta)^m \exp\{N(\tfrac{1}{3}u^3-\zeta u)\}\,du$$

$$= \psi_L(\zeta, N) \qquad\qquad + \phi_L(\zeta, N), \quad \text{say}, \tag{5·6}$$

where the integration is along a finite arc and all the series are absolutely and uniformly convergent. Bounds for ψ_L will now be obtained. We can write

$$\psi_L = \int (u^2-\zeta)^L r_L(u)\exp\{N(\tfrac{1}{3}u^3-\zeta u)\}\,du = N^{-\frac{2}{3}L-\frac{1}{3}}\int (v^2-\eta)^L r_L(N^{-\frac{1}{3}}v)\exp(\tfrac{1}{3}v^3-\eta v)\,dv,$$

where $\eta = N^{\frac{2}{3}}\zeta$, and $r_L(u)$ is regular and bounded in $|u| \leqslant R_u$. The path of integration can be deformed into a curve through one or two saddle points $\pm\eta^{\frac{1}{2}}$. Thus, when $|\arg\eta| < \frac{2}{3}\pi$, the path of integration is along an arc of C_1 passing through the single saddle point $+\eta^{\frac{1}{2}}$, and

$$|\psi_L| < AN^{-\frac{2}{3}L-\frac{1}{3}}\int_{C_1} |v-\eta^{\frac{1}{2}}|^L\{|v-\eta^{\frac{1}{2}}|^L + |\eta|^{\frac{1}{2}L}\}\,|\exp(\tfrac{1}{3}v^3-\eta v)|\,dv.$$

For large η, $|\eta| > R$ say, a bound can be obtained by use of the ideas of the ordinary method of steepest descents if we choose as paths of integration the paths of steepest descent through the saddle point. Thus for $|\eta| > R$

$$|\psi_L| < AN^{-\frac{2}{3}L-\frac{1}{3}}|\eta|^{\frac{1}{4}L-\frac{1}{4}}|\exp(-\tfrac{2}{3}\eta^{\frac{3}{2}})| = AN^{-\frac{1}{2}L-\frac{1}{2}}|\zeta|^{\frac{1}{4}L-\frac{1}{4}}|\exp(-\tfrac{2}{3}\eta^{\frac{3}{2}})|,$$

while for $|\eta| < R$ we have
$$|\psi_L| < AN^{-\frac{2}{3}L-\frac{1}{3}}. \tag{5·7}$$

We also have the expressions (Olver (6), p. 364)

$$\mathrm{Ai}(\eta) \sim A\eta^{-\frac{1}{4}}\exp(-\tfrac{2}{3}\eta^{\frac{3}{2}}), \quad \mathrm{Ai}'(\eta) \sim A\eta^{\frac{1}{4}}\exp(-\tfrac{2}{3}\eta^{\frac{3}{2}})$$

valid for $|\arg\eta| < \frac{2}{3}\pi$, $|\eta| > R$. Thus, for $|\eta| > R$ and bounded ζ,

$$|\psi_L| < AN^{-\frac{1}{2}L-\frac{1}{3}}|\zeta|^{\frac{1}{2}}|\mathrm{Ai}(\eta)| < AN^{-\frac{1}{2}L-\frac{1}{3}}|\mathrm{Ai}(\eta)|$$

and (5·7) shows that a stronger inequality holds for $|\eta| < R$. Similarly,

$$|\psi_L| < AN^{-\frac{1}{2}L-\frac{2}{3}}|\mathrm{Ai}'(\eta)|$$

when $|\arg\eta| < \frac{2}{3}\pi$ and the path of integration is the curve of steepest descent through $\eta^{\frac{1}{2}}$. More care is required when $|\arg(-\eta)| < \frac{2}{3}\pi$, since $\mathrm{Ai}(\eta)$ and $\mathrm{Ai}'(\eta)$ have their zeros along $\arg\eta = \pi$. These zeros cannot coincide, and a simple modification of the argument gives

$$|\psi_L| < AN^{-\frac{1}{2}L-\frac{1}{3}}|\mathrm{Ai}(N^{\frac{2}{3}}\zeta)| + AN^{-\frac{1}{2}L-\frac{2}{3}}|\mathrm{Ai}'(N^{\frac{2}{3}}\zeta)|.$$

Similar bounds can be obtained for ϕ_L defined by equation (5·6) and for contours other than C_1. By choosing L sufficiently large we can make $|\psi_L|$ and $|\phi_L|$ negligible compared with the terms $N^{-2M-\frac{4}{3}}\mathrm{Ai}(N^{\frac{2}{3}}\zeta)$, $N^{-2M-\frac{5}{3}}\mathrm{Ai}'(N^{\frac{2}{3}}\zeta)$ retained in (5·5). This completes the proof of Theorem 2.

6. *Bessel functions.* As an illustration let us consider briefly the function (Watson (7), § 6·2, equation (3))

$$J_N(N\operatorname{sech}\beta) = \frac{1}{2\pi i}\int_{\infty-\pi i}^{\infty+\pi i}\exp\{N(\operatorname{sech}\beta\sinh z - z)\}\,dz, \tag{6·1}$$

when the parameter β is small. Here $f(z,\beta) \equiv \operatorname{sech}\beta\sinh z - z$, $g(z) \equiv 1$. The saddle points are the zeros of $f'(z,\beta) = \operatorname{sech}\beta\cosh z - 1$, i.e. the points $\pm\beta \pm 2m\pi i$, where $m = 0, 1, 2, 3, \ldots$. The relevant saddle points are at $\pm\beta$ (see (7), § 8·31). When $\beta = 0$, two saddle points coincide at $z = 0$, and the theory of § 2 applies, with a multiple of β^2 playing the part of α. The cubic transformation is

$$\operatorname{sech}\beta\sinh z - z = \tfrac{1}{3}u^3 - \zeta(\beta)\,u, \tag{6·2}$$

whence
$$(\operatorname{sech}\beta\cosh z - 1)\frac{dz}{du} = u^2 - \zeta(\beta).$$

The points $z = \pm\beta, u = \pm\zeta^{\frac{1}{2}}$ must correspond, and so, from (6·2),

$$-\tfrac{2}{3}\zeta^{\frac{3}{2}}(\beta) = \tanh\beta - \beta, \quad \zeta \sim 2^{-\frac{2}{3}}\beta^2. \tag{6·3}$$

The theory of § 3 shows that (6·2) has a uniformly regular branch on which z is a regular function of u and β. On this branch the points $z = \pm\beta$, $u = \pm\zeta^{\frac{1}{2}}$ correspond, and the coefficients p_m, q_m of equation (2·2) can be found by repeated differentiation, as indicated in § 2. It then appears that $q_m \equiv 0$, and so z is an odd function of u,

$$\frac{dz}{du} = \sum_0^\infty p_m(\beta)\,(u^2-\zeta)^m, \tag{6·4}$$

when u and β are small. For the Bessel function (6·1) the appropriate contour is C_1, and we obtain

$$\begin{aligned} J_N(N\operatorname{sech}\beta) &\sim \frac{1}{2\pi i}\int_{C_1}\{\Sigma p_m(\beta)\,(u^2-\zeta)^m\}\exp\{N(\tfrac{1}{3}u^3-\zeta u)\}\,du \\ &= \Sigma p_m(\beta)\,F_m(\zeta, N, C_1) \\ &= \frac{\mathrm{Ai}\,(N^{\frac{2}{3}}\zeta)}{N^{\frac{1}{3}}}\Sigma\frac{a_s(\beta)}{N^{2s}} + \frac{\mathrm{Ai}'\,(N^{\frac{2}{3}}\zeta)}{N^{\frac{5}{3}}}\Sigma\frac{b_s(\beta)}{N^{2s}}. \end{aligned} \tag{6·5}$$

We must now find the coefficients $p_m(\beta)$ in (6·4). As has just been said, we can do this in principle by repeatedly differentiating (6·4) and putting $z = \pm\beta$, $u = \pm\zeta^{\frac{1}{2}}$ in the resulting equations. In fact this is a tedious process which will not be discussed at length, since the coefficients $a_s(\beta)$ and $b_s(\beta)$ are already known from the work of Olver (6). We shall calculate only the leading coefficient $p_0(\beta)$. From (6·4), $p_0(\beta)$ is the value of dz/du at $z = \beta$, and from (6·2)

$$(\operatorname{sech}\beta\cosh z - 1)\frac{d^2z}{du^2} + \operatorname{sech}\beta\sinh z\left(\frac{dz}{du}\right)^2 = 2u.$$

Put $z = \beta$, $u = \zeta^{\frac{1}{2}}$, then $\{p_0(\beta)\}^2\tanh\beta = 2\zeta^{\frac{1}{2}}$, and it only remains to fix the sign. Consider dz/du as $z \to 0$, $\beta \to 0$. The order of the limit operations is immaterial, since the regularity is uniform on this branch. Since $z = \pm\beta$, $u = \pm\zeta^{\frac{1}{2}}$ correspond, $dz/du \sim \beta\zeta^{-\frac{1}{2}} \sim 2^{\frac{1}{3}}$ and so $p_0(0) = 2^{\frac{1}{3}}$. But $p_0(\beta)$ is a regular function of β; thus

$$p_0(\beta) = +\left(\frac{2\zeta^{\frac{1}{2}}}{\tanh\beta}\right)^{\frac{1}{2}}.$$

Then we can rewrite (6·5) as

$$J_N(N\operatorname{sech}\beta) \sim \left(\frac{4\zeta(\beta)}{\tanh^2\beta}\right)^{\frac{1}{4}}\left\{\frac{\mathrm{Ai}\,(N^{\frac{2}{3}}\zeta)}{N^{\frac{1}{3}}}\Sigma\frac{A_s(\beta)}{N^{2s}} + \frac{\mathrm{Ai}'\,(N^{\frac{2}{3}}\zeta)}{N^{\frac{5}{3}}}\Sigma\frac{B_s(\beta)}{N^{2s}}\right\}, \tag{6·6}$$

where $A_0(\beta) \equiv 1$. This is precisely Olver's equation (4·24). Olver's results are more general than ours. The validity of (6·6) has here been proved only for β sufficiently small (but independent of N) and for N real, whereas Olver has proved his result for all β in domains extending to infinity and for complex N. He also gives a convenient method for finding $A_s(\beta)$ and $B_s(\beta)$ ((6), § 6) and studies the behaviour of these functions for large ζ ((5), § 9, Lemma 1). There remains the problem of extending the validity of our results to cover the whole region obtained by Olver. The region of small β is nevertheless the most interesting since elsewhere the ordinary method of steepest descents is available.

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