COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL XX, 103–138 (1967)

Uniform Asymptotic Expansion of the Field Scattered by a Convex Object at High Frequencies*

DONALD LUDWIG

1. Introduction

The problem of scattering is concerned with the determination of a function $u_s(x, k)$, which is a solution of the reduced wave equation

$$\Delta u + k^2 u = 0 \,, \tag{1.1}$$

in the exterior of a closed surface S in three dimensions. An incident field $u_i(x, k)$ is prescribed, which satisfies (1.1), and u_s is required to satisfy

$$u_i(x, k) + u_s(x, k) = 0 \text{ for } x \text{ on } S \,. \tag{1.2}$$

The function u_s is also required to satisfy a radiation condition:

$$\lim_{|x|\to\infty} |x| \left(\frac{\partial}{\partial |x|} u_s - iku_s \right) = 0 \,. \tag{1.3}$$

We assume that S has Gauss curvature bounded away from zero (or S is a convex cylinder) and S is analytic. For each positive integer n we give a series in descending powers of k which satisfies the conditions of the problem in an asymptotic sense, i.e., $u_i^{(n)} + u_s^{(n)}$ (consisting of n terms of the series) satisfies the differential equation and boundary condition to arbitrarily high order in k^{-1} if n is chosen large enough. In the region of deep shadow on S (described below), $u_i^{(n)} + u_s^{(n)}$ is bounded by $k^{-2(n-1)/3} \exp\{-k^{1/3}A\}$, where A is positive and proportional to the distance from the shadow boundary.

The asymptotic solution has a complicated form, analogous to an integral representation of the field scattered by a circular cylinder. The various regions involved if S is a convex cylinder are shown in Figure 1.1. In cross-section, two incident rays graze the object S, and their continuations form the shadow boundary, which separates the region of direct illumination (where the reflected

* The research in this paper was supported by the Air Force Cambridge Research Laboratories, Office of Aerospace Research under Contract No. AF 19(628)3868. Reproduction in whole or in part is permitted for any purpose of the United States Government.

field is also present) from the region of shadow. The penumbra is a neighborhood of the shadow boundary (shown in exaggerated form in Figure 1.1) formed by two reflected rays and two rays tangent to S in the shadow region. There is an analogous diagram for a general convex body. In the deeply illuminated region, the solution can be simplified, and we obtain the reflected field predicted by geometrical optics. In fact, this construction leads to a proof of the validity of geometrical optics in this region, since the errors which are made in satisfying

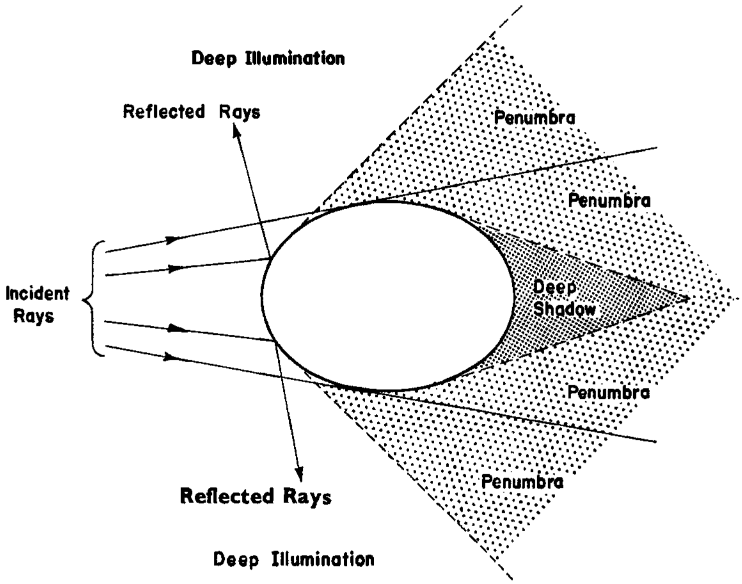

Figure 1.1

the differential equation and boundary condition are smaller than any assigned power of k^{-1}. Such a proof will be given elsewhere. In the deep shadow the solution can be simplified to yield the field predicted by J. B. Keller's geometrical theory of diffraction, [7], [8]. Our procedure does not yield a proof of the geometrical theory of diffraction, since the error in satisfying the boundary condition in the illuminated region is greater than the total field in the shadow. However, this approach does lead to an integral equation (not given here) for the true solution, which might lead to a rigorous theory of diffraction. In the penumbra, which separates the illuminated and shaded regions, we do not simplify the solution, although one might attempt to do so by imitating the treatment of H. M. Nussenzveig [15] for the sphere.

Our method can be generalized to apply to a general impedance boundary condition on S, and to general differential equations with appropriate boundary conditions, e.g., Maxwell's equations in an inhomogeneous, anisotropic medium. Such generalizations will be described in a sequel to this article.

The results of V. I. Fok [5], with modifications described by N. A. Logan and T. S. Yee [12] and the more recent results of E. Zauderer [17] differ from the present results, since they are valid in regions whose size depends upon k.[1] As $k \to \infty$, the region in which Fok's expression is valid shrinks to a single curve, namely the curve of grazing incidence where the shadow boundary begins. As $k \to \infty$, except in special cases, the region in which Zauderer's results are valid shrinks to include only the shaded half of S, together with a portion of the deep shadow region. Although the solutions of Fok and Zauderer can be matched with the direct and reflected waves near the illuminated region, and with the diffracted field near the shaded region, evidently the differential equation is not satisfied in the transition regions. This fact causes difficulty in the attempts of V. S. Buslaev [2] and V. M. Babich [1] to establish the validity of geometrical optics in the illuminated region by using the theory of Fok. There are analogous difficulties in the work of R. Grimshaw [6], who primarily treats the illuminated region. Our methods and results, where they apply to the deep shadow, have much in common with the work of R. M. Lewis, N. Bleistein and D. Ludwig [11] on diffracted waves.

The procedure is based upon a geometrical interpretation of the exact solution for the case of a circular cylinder, which is presented in Appendix A. If r and θ are polar coordinates, and the incident field u_i is a plane wave coming from the left, then, for $0 < \theta < \pi$, u_i can be represented in the form

$$u_i(r\cos\theta, r\sin\theta) = k\int_{-\infty}^{\infty} e^{ik\beta(\pi/2-\theta)} J_{k\beta}(kr)\, d\beta\,, \tag{1.4}$$

where J is the Bessel function. If a is the radius of the cylinder, we have

$$u_s(r\cos\theta, r\sin\theta) = -k\int_{-\infty}^{\infty} e^{ik\beta(\pi/2-\theta)} H^{(1)}_{k\beta}(kr)\, \frac{J_{k\beta}(ka)}{H^{(1)}_{k\beta}(ka)}\, d\beta\,, \tag{1.5}$$

where $H^{(1)}$ is the Hankel function of the first kind. The functions $e^{ik\beta(\pi/2-\theta)}J_{k\beta}(kr)$ and $e^{ik\beta(\pi/2-\theta)}H^{(1)}_{k\beta}(kr)$ are solutions of the reduced wave equation which are, respectively, everywhere regular, and outgoing. The behavior of these functions for large k is given in terms of a family of rays which are tangent to a circle of radius β. Thus in the case of a general convex surface S, we are motivated to represent the incident field in the form

$$u_i(x, k) = k\int_{\alpha_0}^{\alpha_1} e^{ik\theta(x,\alpha)} J(x, \alpha; k)\, d\alpha\,, \tag{1.6}$$

and the scattered field in the form

$$u_s(x, k) = -k\int_{\alpha_0}^{\alpha_1} e^{ik\theta(x,\alpha)} H^{(1)}(x, \alpha; k)\, \frac{\tilde{J}(\alpha;k)}{\tilde{H}^{(1)}(\alpha; k)}\, d\alpha\,. \tag{1.7}$$

[1] *Note added in proofs:* An interesting discussion of the question is given in W. P. Brown [18].

Here $e^{ik\theta(x,\alpha)}J(x, \alpha; k)$ and $e^{ik\theta(x,\alpha)}H^{(1)}(x, \alpha; k)$ are formal asymptotic solutions of the reduced wave equation which are, respectively, everywhere regular, and outgoing. The behavior of these functions for large k is given in terms of a family of rays which are tangent to certain surfaces S_α defined in Appendix B. We have $S_0 = S$, and thus α corresponds to $\beta - a$ in the case of the circular cylinder. Such solutions of the reduced wave equation have been given in the author's paper [13] (see also Yu. A. Kravtsov [9], [10]). Our representations of u_i and u_s are superior to the more usual representation as a superposition of plane waves, since the rays associated with S_α near S are nearly the same as the reflected and diffracted rays near the shadow boundary. The neighborhood of the shadow

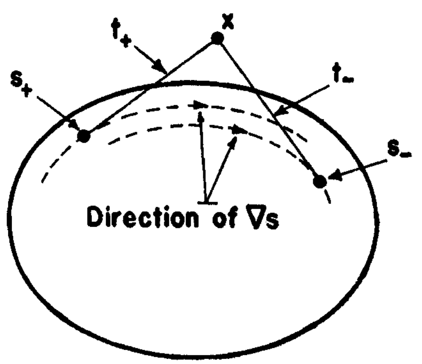

Figure 1.2

boundary is the most crucial region, since geometrical optics and the geometrical theory of diffraction both fail there. As a consequence of the close fitting of the rays, the amplitudes involved in the superposition are regular near $\alpha = 0$ (which corresponds to the shadow boundary). The amplitudes associated with $J(x, \alpha; k)$ and $H^{(1)}(x, \alpha; k)$, the functions $\tilde{J}(\alpha, k)$ and $\tilde{H}^{(1)}(\alpha, k)$, and to a certain extent the surfaces S_α, are determined by requiring that u_i be given by (1.6) and $u_i + u_s$ vanish on S.

According to [13], the functions $\theta(x, \alpha)$ and $J(x, \alpha; k)$ which appear in (1.6) are determined as follows: we let s denote a solution of the surface eikonal equation on S_α, i.e.,

$$(1.8) \qquad (\nabla s)^2 = 1\,, \qquad \nabla s \text{ tangent to } S_\alpha\,.$$

Through each point x_0 on S_α, we draw the straight line which has the direction of $\nabla s(x_0)$. If x is in the exterior of S_α, then in general two such tangents pass through x (see Figure 1.2). The distances from x to the points of tangency are denoted by $t_\pm$, and the values of s at the points of tangency are denoted by $s_\pm$. We define

$$(1.9) \qquad \phi^\pm(x, \alpha) = s_\pm \pm t_\pm\,.$$

It follows that

$$(\nabla\phi^{\pm})^2 = 1 \text{ in the exterior of } S_\alpha\,, \tag{1.10}$$

$$\phi^{\pm} = s \quad \text{on} \quad S_\alpha\,. \tag{1.11}$$

Now we define $\theta(x, \alpha)$ and $\rho(x, \alpha)$ by means of

$$\theta(x, \alpha) = \tfrac{1}{2}(\phi^+(x, \alpha) + \phi^-(x, \alpha))\,, \tag{1.12}$$

$$\rho(x, \alpha) = [\tfrac{3}{4}(\phi^+(x, \alpha) - \phi^-(x, \alpha))]^{2/3}\,. \tag{1.13}$$

It is shown in [13] that $\theta(x, \alpha)$ and $\rho(x, \alpha)$ are regular functions of x; in fact, $\theta(x, \alpha)$ and $\rho(x, \alpha)$ can be determined in the interior of S_α by analytic continuation. We note that there is some arbitrariness in the construction, since the values of s can be prescribed on some curve lying on S_α.

Now $J(x, \alpha; k)$ is defined by

$$J(x, \alpha; k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\{ik(\rho(x, \alpha)\xi - \tfrac{1}{3}\xi^3)\}[\hat{g}(x, \alpha; k) + \xi\hat{h}(x, \alpha; k)]\, d\xi\,. \tag{1.14}$$

Here $\hat{g}$ and $\hat{h}$ are formal series in inverse powers of k; the differential equation (1.1) implies that $\hat{g}$ and $\hat{h}$ satisfy certain transport equations, analogous to the transport equations of geometrical optics (see Appendix B). The series $\hat{g}$ and $\hat{h}$ are completely determined if $\hat{g}$ is specified on S_α (see [13]). Where $\rho > 0$ (in the exterior of S_α), we obtain an asymptotic expansion of (1.14) by the method of stationary phase (see A. Erdélyi [3]). We have

$$e^{ik\theta(x,\alpha)}J(x, \alpha; k) \sim e^{ik\phi^+(x,\alpha)}\hat{z}_+(x, \alpha; k) + e^{ik\phi^-(x,\alpha)}\hat{z}_-(x, \alpha; k)\,, \tag{1.15}$$

where $\hat{z}_+$ and $\hat{z}_-$ are series in descending powers of k. The expansion (1.15) is the analogue of the Debye expansion of the Bessel function.

In Section 2 we shall ensure that (1.6) is satisfied to arbitrarily high order in k^{-1} by stationary phase evaluation of a corresponding double integral. The resulting conditions determine $\theta(x, \alpha)$ (i.e., s) and $\hat{g}(x, \alpha)$ on a curve C_α on S_α, where the incident rays are tangent to S_α.

In Section 3 we attempt a representation of $u_s(x, k)$ in the form (1.7). The function $H^{(1)}(x, \alpha; k)$ (the analogue of the Hankel function) is defined by

$$H^{(1)}(x, \alpha; k) = \frac{1}{\pi}\int_L \exp\{ik(\rho(x, \alpha)\xi - \tfrac{1}{3}\xi^3)\}[g(x, \alpha; k) + \xi h(x, \alpha; k)]\, d\xi\,, \tag{1.16}$$

where the integration is taken over the contour L shown in Figure 1.3. The functions $\rho(x, \alpha)$ and $\theta(x, \alpha)$ are given by (1.12), (1.13), and the requirement that $e^{ik\theta(x,\alpha)}H^{(1)}(x, \alpha; k)$ be a solution of (1.1) implies certain transport equations for

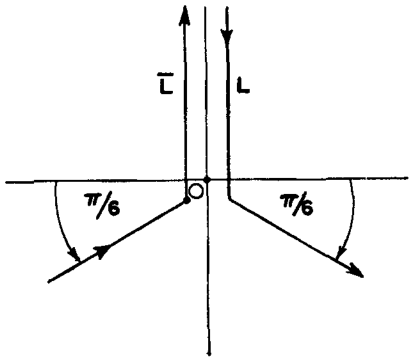

Figure 1.3

g and h; g and h are completely determined if g is specified on S_α. Furthermore, the contour L has been chosen in such a way that by applying the method of steepest descent (see [3]) for $\rho > 0$ and k large we obtain

$$e^{ik\theta}H^{(1)}(x, \alpha; k) \sim 2e^{ik\phi^+(x,\alpha)}z_+(x, \alpha; k) . \tag{1.17}$$

The amplitude z_+ is specified in (3.17), (3.18). From the definition (1.9) of ϕ^+, we see that $\phi^+ \sim |x|$ for $|x|$ large. It follows that u_s satisfies the radiation condition (1.3). The details of the behavior of u_s at infinity are not crucial to our present purpose. They will be given elsewhere. The functions $\tilde{J}(\alpha, k)$ and $\tilde{H}^{(1)}(\alpha, k)$ are defined in a manner analogous to (1.14) and (1.16):

$$\tilde{J}(\alpha, k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\{ik(\tilde{\rho}(\alpha)\xi - \tfrac{1}{3}\xi^3)\}\, d\xi , \tag{1.18}$$

$$\tilde{H}^{(1)}(\alpha, k) = \frac{1}{\pi}\int_L \exp\{ik(\tilde{\rho}(\alpha)\xi - \tfrac{1}{3}\xi^3)\}\, d\xi . \tag{1.19}$$

In fact, $\tilde{J}$ and $\tilde{H}^{(1)}$ are expressible in terms of the Airy function (see [3]). In Section 3 the function $\tilde{\rho}(\alpha)$ will be determined, and $g(x, \alpha)$ will be specified on a certain curve R_α on S_α, by requiring that the boundary condition (1.2) be satisfied on the illuminated side of S. At the same time, the calculation will show that, in the illuminated region, u_s behaves like the reflected wave predicted by geometrical optics.

Up to this point, we have required only that the family of surfaces S_α be nested, convex and analytic, and the amplitudes g and $\hat{g}$ have each been specified on a single curve. If we require in addition that $g(x, \alpha; k) = \hat{g}(x, \alpha; k)$ and $h(x, \alpha; k) = \hat{h}(x, \alpha; k)$ if x is in the shadow of S, then $u_i + u_s$ vanishes in the shadow to the approximation of geometrical optics, and $u_i(x) + u_s(x)$ can be written as a sum of residues. The residues correspond to values of α of the order of $k^{-2/3}$; thus the requirement that the residues shall vanish on S imposes conditions on the integrands of (1.6) and (1.7) for α near zero. In particular, the surfaces S_α are determined asymptotically for α near zero, and the amplitudes $g(x, \alpha; k)$ and $h(x, \alpha; k)$ must satisfy surface transport equations on the shaded side of S, for α near 0. Finally, each term of the residue sum can be interpreted as a creeping or diffracted wave, and our results agree with J. B. Keller's geometrical theory of diffraction, [8]. The details of the preceding discussion, given in Section 4, are analogous to the treatment of creeping waves in [11]. However, in the present work the creeping waves appear together with the appropriate diffraction coefficients, with no need for additional assumptions.

Sections 2, 3 and 4 are devoted primarily to listing conditions on the components of our "Ansatz" (1.6) and (1.7), and deriving their consequences. Section 5, with the associated Appendix B, provides a construction of the functions which appear in (1.6) and (1.7), and a justification of certain formal procedures used earlier. The treatment of Sections 2–5 thus yields a formal asymptotic series which satisfies the differential equation (1.1), and which satisfies the boundary condition (1.2) in the deeply illuminated region and the deep shadow. The verification that (1.2) is also satisfied in the penumbra is given in Section 6, using the results of Appendix C and of the author's paper [14]. This latter part of the analysis has no analogue in the treatment of the circular cylinder in Appendix A, since the boundary condition is satisfied exactly in that case. Certain of our methods in Section 6 are related to boundary layer techniques, but the various regions are introduced after the solution has been constructed, so that there is no question of matching of solutions.

The author is indebted to J. B. Keller for drawing his attention to the problem of diffraction and generously making available his knowledge of the subject. The author is also indebted to R. M. Lewis, N. Bleistein and J. Moser for a number of fruitful discussions of the problem, and to J. Cohen for reviewing the manuscript.

2. Representation of the Incident Field

As outlined in the introduction, we attempt to represent the incident field in the form (1.6), with

$$u_i(x, k) = e^{ix\phi_i(x)} z_i(x, k) ,$$

where $z_i(x, k)$ is a formal series in integral powers of k^{-1}. We require (1.6) to be valid in a neighborhood of the shadow boundary C on S. The results are summarized at the end of this section.

Using the definition (1.14) of $J(x, \alpha; k)$, we may rewrite (1.6) as a double integral:

$$e^{ik\phi_i(x)} z_i(x; k) = \frac{k}{2\pi} \iint \exp \{ik(\theta(x, \alpha) + \rho(x, \alpha)\xi - \tfrac{1}{3}\xi^3)\} \times [\hat{g}(x, \alpha; k) + \xi \hat{h}(x, \alpha; k)]\, d\alpha\, d\xi . \tag{2.1}$$

This integral can be evaluated asymptotically for large k by two-dimensional stationary phase; thus (1.6) is equivalent to

$$e^{ik\phi_i(x)} z_i(x; k) = \sum e^{ik\hat{\phi}(x,\hat{\alpha},\hat{\xi})} \hat{z}(x; k) , \tag{2.2}$$

where

$$\hat{\phi}(x, \alpha, \xi) = \theta + \rho\xi - \tfrac{1}{3}\xi^3 , \tag{2.3}$$

and $\hat{z}(x, k)$ is a formal series in descending powers of k, involving derivatives of $\hat{\phi}$, $\hat{g}$ and $\hat{h}$ at $(x, \hat{\alpha}, \hat{\xi})$. The summation in (2.2) is taken over all pairs $(\hat{\alpha}, \hat{\xi})$, depending upon x, which satisfy

$$\frac{\partial}{\partial\alpha} \hat{\phi}(x, \hat{\alpha}, \hat{\xi}) = \theta_\alpha(x, \hat{\alpha}) + \hat{\xi}\rho_\alpha(x, \hat{\alpha}) = 0 , \tag{2.4}$$

$$\frac{\partial}{\partial\xi} \hat{\phi}(x, \hat{\alpha}, \hat{\xi}) = \rho(x, \hat{\alpha}) - \hat{\xi}^2 = 0 . \tag{2.5}$$

In the present case, only one stationary point will appear. We shall show in Section 5 that there is no contribution from the endpoints of the interval of integration. We regard (2.2) as a condition on θ, ρ and $\hat{z}$ to be satisfied at the stationary point. This condition will hold if we set

$$\phi_i(x) = \hat{\phi}(x, \hat{\alpha}, \hat{\xi}) , \tag{2.6}$$

$$z_i(x; k) = \hat{z}(x; k) . \tag{2.7}$$

Conditions (2.4)–(2.6) can be replaced by an equivalent set of conditions involving $\phi^\pm(x, \alpha)$. From (2.6), (2.5) and (1.12), (1.13) we have

$$\phi_i(x) = \hat{\phi}(x, \hat{\alpha}, \hat{\xi}) = \phi^\pm(x, \hat{\alpha}) , \tag{2.8}$$

and (2.4) becomes

$$\frac{\partial}{\partial\alpha} \phi^\pm(x, \hat{\alpha}) = 0 . \tag{2.9}$$

By differentiating (2.8) and applying (2.9), we obtain

$$\nabla\phi_i(x) = \nabla\phi^\pm(x, \hat{\alpha}) . \tag{2.10}$$

Thus $\nabla\phi^\pm(x, \hat{\alpha})$ must be constant along incident rays. On the other hand, for fixed α, $\nabla\phi^\pm$ is constant along rays which are tangent to C_α. We conclude that if (2.4)–(2.6) or (2.8), (2.9) are to be satisfied, the incident ray through x must be tangent to the surface $S_{\hat{\alpha}}$. This condition determines $\hat{\alpha}$ as a function of x.

In order to ensure that (2.8) is satisfied, we define C_α as the locus of points on S_α where the incident ray direction $\nabla\phi_i$ is tangent to S_α. On C_α we specify (see (1.8)–(1.12))

$$s = \theta(x, \alpha) = \phi_i(x)\,, \tag{2.11}$$

and we choose the direction of ∇s so that

$$\nabla s = \nabla\theta(x, \alpha) = \nabla\phi_i(x) \qquad \text{on} \qquad C_\alpha\,. \tag{2.12}$$

These relations, together with (1.8), specify s completely on S_α, and ultimately determine $\theta(x, \alpha)$ and $\rho(x, \alpha)$ everywhere.

We first note that, with this choice of $\theta(x, \alpha)$, (2.8)–(2.9) are satisfied on C_α, with $\hat{\alpha} = \alpha$. In fact, setting $\hat{\alpha} = \alpha$, we have $\phi^+(x, \alpha) = \phi^-(x, \alpha) = \theta(x, \alpha)$ (see (1.11), (1.12)) and hence (2.8) is satisfied if x is on C_α. We let $y(\alpha)$ be a point on C_α and we differentiate (2.8) with respect to α:

$$\frac{dy}{d\alpha}\cdot\nabla\phi^\pm(y(\alpha), \alpha) + \phi_\alpha^\pm(y(\alpha), \alpha) = \frac{dy}{d\alpha}\cdot\nabla\phi_i(y(\alpha))\,. \tag{2.13}$$

In view of (2.12), (2.13) implies (2.9) on C_α.

Now it remains to be shown that (2.8) is satisfied everywhere (in a neighborhood of S) if it is satisfied on C_α. By differentiating (1.10) with respect to α, we see that $\phi_\alpha^\pm$ is constant along the rays associated with $\phi^\pm$. Since (2.9) holds for x on C_α, (2.9) also holds at $(x, \hat{\alpha}(x))$. Since (2.8) and (2.10) are satisfied at $C_{\hat{\alpha}}$ and both ϕ^i and $\phi^\pm(x, \hat{\alpha})$ satisfy the eikonal equation (1.10), (2.8) is satisfied at $(x, \hat{\alpha}(x))$.

We observe that the incident phase is equal to $\phi^-(x, \hat{\alpha})$ if x is on the portion of the incident ray which has not yet reached $S_{\hat{\alpha}}$, and the incident phase is equal to $\phi^+(x, \hat{\alpha})$ if x is on the portion of the incident ray which has left S_α.

In order to determine $\hat{z}(x, k)$ (mentioned in (2.2) and (2.7)), we first verify that the integral in (2.1) has a simple stationary point at $\hat{\alpha}$, $\hat{\xi}$, i.e.,

$$\Delta = \begin{vmatrix} \hat{\phi}_{\alpha\alpha} & \hat{\phi}_{\alpha\xi} \\ \hat{\phi}_{\alpha\xi} & \hat{\phi}_{\xi\xi} \end{vmatrix} = \begin{vmatrix} \theta_{\alpha\alpha} + \rho_{\alpha\alpha}\hat{\xi} & \rho_\alpha \\ \rho_\alpha & -2\hat{\xi} \end{vmatrix} \neq 0\,. \tag{2.14}$$

On C_α, we have $\hat{\alpha} = \alpha$, $\hat{\xi} = 0$, and thus

$$\Delta = -(\rho_\alpha(x, \alpha))^2 \text{ if } x \text{ is on } C_\alpha\,. \tag{2.15}$$

Since the surfaces S_α are nested, we have $\rho_\alpha \neq 0$ (see also Appendix B), and thus $\Delta \neq 0$ on C_α. A study of the geometry of the rays would show that, in fact, $\Delta \neq 0$ everywhere.

In order to see that $\hat{g}(x, \alpha; k)$ is determined on C_α by (2.7), we need some additional details about $\hat{g}$ and $\hat{z}$: we write

$$(2.16) \qquad \hat{g}(x, \alpha; k) = \sum_{j=0}^{\infty} \hat{g}_j(x, \alpha)(ik)^{-j},$$

$$(2.17) \qquad \hat{h}(x, \alpha; k) = \sum_{j=0}^{\infty} \hat{h}_j(x, \alpha)(ik)^{-j}.$$

Then $\hat{z}(x, k)$ will have the form

$$(2.18) \qquad \hat{z}(x, k) = \sum_{j=0}^{\infty} \hat{z}_j(x)(ik)^{-j},$$

where

$$(2.19) \qquad \hat{z}_j(x) = \frac{\hat{g}_j(x, \hat{\alpha}) + \hat{\xi}\hat{h}_j(x, \hat{\alpha})}{\sqrt{|\Delta|}} + f_j ;$$

f_j involves $\hat{g}_0, \cdots, \hat{g}_{j-1}, \hat{h}_0, \cdots, \hat{h}_{j-1}$ and their derivatives. Thus by equating coefficients of corresponding powers of k in (2.7), we obtain a system of equations which can be solved recursively. In particular, on C_α we have $\hat{\xi} = 0$, $\hat{\alpha} = \alpha$, and $\hat{g}(x, \alpha; k)$ is determined from (2.7). In view of the transport equations satisfied by z_i and $|\Delta|^{-1/2}(\hat{g} + \hat{\xi}\hat{h})$, we conclude that (2.7) is satisfied everywhere.

In summary, we see that (1.6) can be satisfied to arbitrary order in k^{-1} if $\theta(x, \alpha) = s$ and $\hat{g}(x, \alpha; k)$ are appropriately specified on C_α (see (2.11), (2.12), (2.7) and (2.16)–(2.19)). C_α is defined as the locus on S_α where the incident rays are tangent to S_α. We remark that the representation (1.6) is a generalization of the Kantorovich-Lebedev transform of the incident field (see A. Erdélyi *et al.* [4]).

3. The Boundary Condition on the Illuminated Side of S

As described in the introduction, we represent the scattered field in the form (1.7); the integrand in (1.7) is described in (1.16)–(1.19). We shall determine $\tilde{\rho}(\alpha)$ (which appears in the definitions of $\tilde{J}(\alpha)$ and $\tilde{H}^{(1)}(\alpha)$) by applying the method of stationary phase to the integral (1.7), and matching the incident phase and the phase of u_s on the illuminated side of S. The amplitude of $H^{(1)}(x, \alpha; k)$ is specified at the stationary point (and hence on a certain curve R_α on S) by matching the amplitudes of u_i and u_s. It is easy to verify that our procedure yields the reflected wave in the deeply illuminated region, and that the boundary condition (1.2) is satisfied in this region.

We first restrict the integration in (1.7) to an interval where $\alpha < 0$. In this case $\rho(x, \alpha) > 0$ if x is outside S, and $H^{(1)}(x, \alpha; k)$ may be replaced by its asymptotic expansion (1.17). We also assume that $\tilde{\rho}(\alpha) > 0$, and thus $\tilde{J}(\alpha, k)$ and

$\tilde{H}^{(1)}(\alpha, k)$ may be replaced by their asymptotic expansions:

$$\tilde{J}(\alpha, k) \sim e^{ik\psi^{+}(\alpha)}\tilde{z}_{+}(\alpha; k) + e^{ik\psi^{-}(\alpha)}\tilde{z}_{-}(\alpha, k) , \tag{3.1}$$

$$\tilde{H}^{(1)}(\alpha, k) \sim 2e^{ik\psi^{+}(\alpha)}\tilde{z}_{+}(\alpha, k) . \tag{3.2}$$

Here

$$\psi^{\pm}(\alpha) = \pm\tfrac{2}{3}(\tilde{\rho}(\alpha))^{3/2} , \tag{3.3}$$

and $\tilde{z}_{\pm}$ are given in terms of $\tilde{\rho}(\alpha)$ (see (3.18), (3.19)). In Appendix B we shall verify that $\tilde{\rho}(\alpha) > 0$ if $\alpha < 0$; the interval of integration where α is near zero or

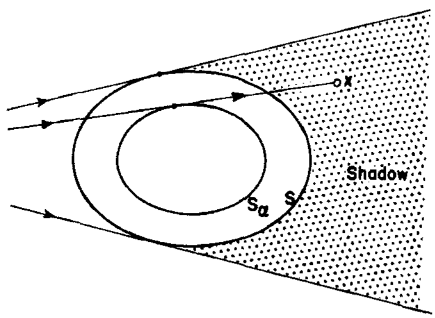

Figure 3.1

$\alpha > 0$ will be treated in Section 6. It is shown in Section 5 that there is no contribution from the endpoint $\alpha = \alpha_0$. Inserting (3.1)–(3.3) into (1.7), we obtain

$$\begin{aligned} u_s(x, k) \sim & -k\int e^{ik\phi^{+}(x,\alpha)} z_{+}(x, \alpha; k)\, d\alpha \\ & -k\int e^{ik[\phi^{+}(x,\alpha)+\psi^{-}(\alpha)-\psi^{+}(\alpha)]} z_{+}(x, \alpha; k)\, \frac{\tilde{z}_{-}(\alpha, k)}{\tilde{z}_{+}(\alpha, k)}\, d\alpha . \end{aligned} \tag{3.4}$$

The first integral will have stationary points only where $\phi_{\alpha}^{+}(x, \alpha) = 0$. The discussion of the previous section shows that $\phi_{\alpha}^{+}(x, \alpha) = 0$ only if the ray through x is tangent to S_α (at C_α), and x is on the outgoing half of the ray (since ϕ^{+} corresponds to an outgoing wave). Since $\alpha < 0$, S_α is inside S, and we conclude that $\phi_{\alpha}^{+}(x, \alpha) = 0$ only if x is in the shadow of S (see Figure 3.1). Hence the first integral is smaller than any power of k^{-1} in the deeply illuminated region.

Stationary phase evaluation of the second integral yields

$$u_s(x, k) \sim -e^{ik\tilde{\phi}(x,\tilde{\alpha})} z_s(x, k) , \tag{3.5}$$

where

$$\tilde{\phi}(x, \alpha) = \phi^{+}(x, \alpha) + \psi^{-}(\alpha) - \psi^{+}(\alpha) ; \tag{3.6}$$

the amplitude $z_s(x, k)$ will be discussed later (see (3.17)–(3.22)). The value of α at the stationary point (denoted by $\tilde{\alpha}(x)$) is determined from the condition of stationary phase:

$$\tilde{\phi}_\alpha(x, \tilde{\alpha}) = \phi_\alpha^+(x, \tilde{\alpha}) + \psi_\alpha^-(\tilde{\alpha}) - \psi_\alpha^+(\tilde{\alpha}) = 0 . \tag{3.7}$$

The boundary condition (1.2) states that $u_i + u_s$ shall vanish on S: thus we must have

$$\phi_i(x) = \tilde{\phi}(x, \tilde{\alpha}) \text{ if } x \text{ is on } \Gamma_i . \tag{3.8}$$

Here Γ_i denotes the illuminated side of S. Differentiating (3.8) in a tangential direction and using (3.7), we see that we must have

$$\nabla\phi_i \cdot dx = \nabla\tilde{\phi}(x, \tilde{\alpha}) \cdot dx \quad \text{if} \quad x \in \Gamma_i , \qquad dx \text{ tangent to } \Gamma_i . \tag{3.9}$$

Equation (3.9), together with the conditions that $(\nabla\tilde{\phi})^2 = 1$, and the direction of $\nabla\tilde{\phi}$ is outgoing, determines $\nabla\tilde{\phi}$ uniquely: $\nabla\tilde{\phi}$ has the direction of the reflected ray at x on Γ_i. Now we determine $\tilde{\alpha}(x)$ if x is on Γ_i by requiring that the reflected ray at x, continued inside S, be tangent to $S_{\tilde{\alpha}}$, i.e., $\tilde{\alpha}(x)$ is the minimum of the indices of the surfaces S_α which are intersected by the reflected ray through x. Since the surfaces S_α are convex, $\tilde{\alpha}$ is a smooth function of x.

To determine $\tilde{\rho}$ (as a function of x) on Γ_i we use (3.3), (3.6) and (3.8):

$$\tfrac{4}{3}(\tilde{\rho})^{3/2} = \phi^+(x, \tilde{\alpha}) - \phi_i(x) , \tag{3.10}$$

or equivalently,

$$\psi^+(\tilde{\alpha}) - \psi^-(\tilde{\alpha}) = \phi^+(x, \alpha) - \phi_i(x) . \tag{3.11}$$

Before making this definition of $\tilde{\rho}$, we should verify that $\phi^+(x, \tilde{\alpha}) - \phi_i(x) > 0$ on Γ_i. However, we omit the verification, since it is a consequence of the more elaborate discussion of Appendix C. In order to show that $\tilde{\rho}$, defined by (3.10), is actually a function of $\tilde{\alpha}$, we differentiate (3.10) in a direction tangential to Γ_i:

$$2\sqrt{\tilde{\rho}}\, d\tilde{\rho} = \nabla\phi^+(x, \tilde{\alpha}) \cdot dx - \nabla\phi_i(x) \cdot dx + \phi_\alpha^+(x, \tilde{\alpha})\, d\tilde{\alpha} . \tag{3.12}$$

In view of (3.9), $\tilde{\rho}$ is constant if $\tilde{\alpha}$ is constant, i.e., $\tilde{\rho}$ can be determined as a function of $\tilde{\alpha}$ alone. We define $\tilde{\rho}(\alpha)$ for $\alpha > 0$ by analytic continuation.

In order to verify (3.7), we differentiate (3.11) in a tangential direction and apply (3.9):

$$\psi_\alpha^+(\tilde{\alpha})\, d\tilde{\alpha} - \psi_\alpha^-(\tilde{\alpha})\, d\tilde{\alpha} = \phi_\alpha^+(x, \tilde{\alpha})\, d\tilde{\alpha} . \tag{3.13}$$

Thus (3.7) is satisfied if $d\tilde{\alpha} \neq 0$, which is shown in Appendix C, (C.17). In order to compute $\tilde{\phi}_{\alpha\alpha}(x, \tilde{\alpha})$, we differentiate (3.7) in a tangential direction:

$$\nabla\phi_\alpha^+(x, \tilde{\alpha}) \cdot dx + \tilde{\phi}_{\alpha\alpha}(x, \tilde{\alpha})\, d\tilde{\alpha} = 0 . \tag{3.14}$$

Hence $\tilde{\phi}_{\alpha\alpha}(x, \tilde{\alpha}) \neq 0$ if $\nabla\phi_\alpha^+$ is not normal to Γ_i, which is shown in Appendix C, (C.15).

In order to determine the amplitude $z_+(x, \tilde{\alpha})$, we examine the coefficients in (3.5) more closely. The amplitudes associated with $H^{(1)}(x, \alpha; k)$ (see (1.16)) have the form

$$g(x, \alpha; k) = \sum_{j=0}^{\infty} g_j(x, \alpha)(ik)^{-j}, \tag{3.15}$$

$$h(x, \alpha; k) = \sum_{j=0}^{\infty} h_j(x, \alpha)(ik)^{-j}. \tag{3.16}$$

Consequently, the amplitude in the asymptotic expansion (1.17) of $e^{ik\theta}H^{(1)}(x, \alpha; k)$ has the form

$$z_+(x, \alpha; k) = \sum_{j=0}^{\infty} z_{+j}(x, \alpha)(ik)^{-j}, \tag{3.17}$$

where

$$z_{+j}(x, \alpha) = \sqrt{\frac{2\pi}{k}}\, e^{-i\pi/4}\, \frac{g_j(x, \alpha) + \sqrt{\rho(x, \alpha)}\,h_j(x, \alpha)}{\sqrt[4]{\rho(x, \alpha)}} + f_j\,; \tag{3.18}$$

here f_j involves g_l, h_l and their derivatives for $l < j$. Similarly, the amplitudes in the asymptotic expansions (3.1) and (3.2) of $\tilde{J}(\alpha, k)$ and $\tilde{H}^{(1)}(\alpha, k)$ have the form

$$\tilde{z}_\pm(\alpha, k) = \sum_{j=0}^{\infty} z_{\pm j}(\alpha)(ik)^{-j}, \tag{3.19}$$

where

$$\tilde{z}_{\pm 0} = \sqrt{\frac{2\pi}{k}}\, e^{\mp i\pi/4}\, \frac{1}{\sqrt[4]{\tilde{\rho}(\alpha)}}. \tag{3.20}$$

Inserting (3.17)–(3.20) into (3.4) and applying the method of stationary phase, we obtain

$$z_s(x, k) = \sum_{j=0}^{\infty} z_{sj}(x)(ik)^{-j}, \tag{3.21}$$

where

$$z_{sj}(x) = \frac{1}{\sqrt{|\tilde{\phi}_{\alpha\alpha}(x, \tilde{\alpha})|}}\, \frac{g_j(x, \tilde{\alpha}) + \sqrt{\rho(x, \tilde{\alpha})}\, h_j(x, \tilde{\alpha})}{\sqrt[4]{\rho(x, \tilde{\alpha})}} + F_j\,. \tag{3.22}$$

Here F_j involves g_l, h_l and their derivatives for $l < j$. Since $\tilde{\phi}(x, \tilde{\alpha}) = \phi_i(x)$ for x on Γ_i, $u_i(x, k) + u_s(x, k)$ will vanish on Γ_i if we set

$$z_s(x, k) = z_i(x, k) \text{ for } x \text{ on } \Gamma_i\,. \tag{3.23}$$

In view of (3.21), (3.22), the amplitudes $g_j(x, \tilde{\alpha}) + \sqrt{\rho(x, \tilde{\alpha})}\, h_j(x, \tilde{\alpha})$ can be determined recursively from (3.23).

Finally, we consider points x in the deeply illuminated region, which are not necessarily on Γ_i. From (3.7), it follows that $\tilde{\alpha}$ is constant where $\phi_\alpha^+(x, \tilde{\alpha})$ is constant; on the other hand, by differentiating (1.10) with respect to α, we see that ϕ_α^+ is constant along the reflected rays. Since $\tilde{\phi}(x, \tilde{\alpha}) = \phi_i(x)$ if x is on Γ_i (see (3.8)), we conclude that $\tilde{\phi}(x, \tilde{\alpha})$ is the reflected phase. In a similar fashion, it follows that $-z_s(x, k)$ is the amplitude associated with the reflected wave.

4. The Boundary Condition on the Shaded Side of S

As outlined in the introduction, we shall rewrite the total field $u_i(x, k) + u_s(x, k)$ as a sum of residues when x is in the deep shadow. We require that the residues vanish when x is on Γ_s, the shaded side of S. This requirement determines the behavior of $\rho(x, \alpha)$ for x on S and α near zero (ultimately it determines the surface S_α for α near zero); the amplitudes of $H^{(1)}(x, \alpha; k)$ and $J(x, \alpha; k)$ are also determined for α near zero. In the next section and Appendix B we shall construct $g(x, \alpha; k)$, $h(x, \alpha; k)$, $\hat{g}(x, \alpha; k)$ and $\hat{h}(x, \alpha; k)$, and we shall justify the transformation of the integral into a sum of residues. The results are summarized at the end of this section.

Motivated by the procedure in the case of the circular cylinder (outlined in Appendix A), we write the sum $u_i + u_s$ as a single integral:

$$u_i(x, k) + u_s(x, k) = k\int_{\alpha_0}^{\alpha_1} e^{ik\theta(x,\alpha)}\left[J(x, \alpha; k) - H^{(1)}(x, \alpha; k)\,\frac{\tilde{J}(\alpha, k)}{\tilde{H}^{(1)}(\alpha, k)}\right] d\alpha\,. \tag{4.1}$$

The next step requires the identity

$$J(x, \alpha; k) = \tfrac{1}{2}[H^{(1)}(x, \alpha; k) + H^{(2)}(x, \alpha; k)]\,, \tag{4.2}$$

where

$$H^{(2)}(x, \alpha; k) = \frac{1}{\pi}\int_{\bar{L}} \exp\{ik[\rho(x, \alpha)\xi - \tfrac{1}{3}\xi^3]\}[g(x, \alpha; k) + \xi h(x, \alpha; k)]\, d\xi\,. \tag{4.3}$$

The contour $\bar{L}$ is shown in Figure 1.3. There is a similar definition of $\tilde{H}^{(2)}(\alpha, k)$. In view of the definitions of J and $H^{(1)}$, the validity of (4.2) requires only that

$$g(x, \alpha; k) = \hat{g}(x, \alpha; k)\,, \tag{4.4}$$

$$h(x, \alpha; k) = \hat{h}(x, \alpha; k)\,, \tag{4.5}$$

i.e., that the amplitudes of $J(x, \alpha; k)$ and $H^{(1)}(x, \alpha; k)$ agree. For our present purposes, it suffices that (4.4), (4.5) (and hence (4.2)) hold if x is in the shadow of S. In view of the definitions of $\tilde{J}(\alpha, k)$ and $\tilde{H}^{(1)}(\alpha, k)$, we also have

$$\tilde{J}(\alpha, k) = \tfrac{1}{2}[\tilde{H}^{(1)}(\alpha, k) + \tilde{H}^{(2)}(\alpha, k)]\,. \tag{4.6}$$

Substituting (4.2) and (4.6) into (4.1), we obtain

$$(4.7)\quad u_i(x,k) + u_s(x,k) = \frac{k}{2}\int_{\alpha_0}^{\alpha_1} e^{ik\theta(x,\alpha)} \times \left[H^{(2)}(x,\alpha;k) - H^{(1)}(x,\alpha;k)\,\frac{\tilde{H}^{(2)}(\alpha,k)}{\tilde{H}^{(1)}(\alpha,k)}\right] d\alpha\,.$$

We remark that an evaluation of the integral (4.1) by stationary phase would show that in the shadow the term which is removed from u_i and u_s is equal to the incident field (to the approximation of geometrical optics). Thus the incident and scattered fields cancel in the shadow to the approximation of geometrical optics. Indeed, an evaluation of (4.7) by the method of stationary phase would show that there are no real stationary points, i.e., the integral is smaller than any power of k^{-1}. There are no endpoint contributions because of the way in which the integrands are defined in Section 5.

We observe that the integrand in (4.7) has poles at the zeros of $\tilde{H}^{(1)}(\alpha, k)$, which lie in the upper half-plane. We deform the path of integration of (4.7) into the upper half-plane and obtain a sum of residues at these poles. The justification of this step will be given in Section 5. Thus we obtain

$$(4.8)\qquad u_i(x,k) + u_s(x,k) \sim \sum_{l=0}^{L} u_l(x,k)\,,$$

where

$$(4.9)\qquad u_l(x,k) = i\pi k e^{ik\theta(x,\alpha_l)} H^{(1)}(x,\alpha_l\,;k)\,\frac{\tilde{H}^{(2)}(\alpha_l\,,k)}{\dfrac{\partial}{\partial\alpha}\tilde{H}^{(1)}(\alpha_l\,,k)}\,,$$

and α_l are functions of k which satisfy

$$(4.10)\qquad \tilde{H}^{(1)}(\alpha_l\,,k) = 0\,.$$

According to the definition (1.19) of $\tilde{H}^{(1)}(\alpha, k)$, we have

$$(4.11)\qquad \tilde{H}^{(1)}(\alpha,k) = 2k^{1/3}e^{i\pi/3}A(-k^{2/3}e^{2\pi i/3}\tilde{\rho}(\alpha))\,,$$

where A is the Airy function (see [3]). To lowest order in k^{-1}, (4.10), (4.11) imply that

$$(4.12)\qquad \tilde{\rho}(\alpha_l) = -k^{-2/3}e^{-2\pi i/3}q_l\,,$$

where q_l is a zero of the Airy function. It is known (see [3]) that these roots are all real and negative. We recall that $\tilde{\rho}(\alpha)$ is given by (3.10). Since $\tilde{\alpha}(x) = 0$ and x on S implies that x is on C (the shadow boundary), and since $\phi^+(x, 0) = \phi^-(x, 0) = \phi_i(x)$ if x is on C (see (1.9) and (2.11)), we conclude from (3.10) that

$$\tilde{\rho}(0) = 0\,. \tag{4.13}$$

Thus from (4.12) and (4.13), to a first approximation, we have

$$\alpha_l \sim -k^{-2/3}\frac{e^{-2\pi i/3}q_l}{\tilde{\rho}_\alpha(0)}\,, \tag{4.14}$$

i.e., α_l has order $k^{-2/3}$.

In order to impose the boundary condition that u_l shall vanish for x on Γ_s (the shaded side of S), we rewrite (4.9), expressing $H^{(1)}(x, \alpha; k)$ in terms of the Airy function:

$$\begin{aligned} u_l(x, k) &= k^{2/3}e^{5\pi i/6}\frac{\tilde{H}^{(2)}(\alpha_l\,, k)}{\dfrac{\partial}{\partial\alpha}\tilde{H}^{(1)}(\alpha_l\,, k)}e^{ik\theta(x,\alpha_l)} \\ &\times [A(-k^{2/3}e^{2\pi i/3}\rho(x, \alpha_l))g(x, \alpha_l\,; k) + ik^{-1/3}A'(-k^{2/3}e^{2\pi i/3}\rho(x, \alpha_l))h(x, \alpha_l\,; k)]\,. \end{aligned} \tag{4.15}$$

Consequently, u_l will vanish to highest order in k for x on Γ_s if we require that

$$A(-k^{2/3}e^{2\pi i/3}\rho(x, \alpha_l)) = 0 \text{ for } x \text{ on } \Gamma_s\,. \tag{4.16}$$

Comparing (4.11), (4.12) and (4.16), we must have at least to first order

$$\rho(x, \alpha_l) \sim \tilde{\rho}(\alpha_l) \text{ for } x \text{ on } \Gamma_s\,. \tag{4.17}$$

In Appendix B, we construct S_α for α near 0 in such a way that (see (C.1))

$$\rho(x, \alpha) - \tilde{\rho}(\alpha) = O(\alpha^n) \text{ for } x \text{ on } S\,. \tag{4.18}$$

Moreover (see (B.3))

$$\rho_\alpha(x, 0) = -1\;; \tag{4.19}$$

hence

$$\tilde{\rho}_\alpha(0) = -1\,. \tag{4.20}$$

Combining (4.14) and (4.18), we obtain

$$\rho(x, \alpha_l) - \tilde{\rho}(\alpha_l) = O(k^{-2n/3}) \text{ for } x \text{ on } S\,. \tag{4.21}$$

It follows from (4.12) and (4.21) that

$$A(-k^{2/3}e^{2\pi i/3}\rho(x, \alpha_l) = O(k^{-2(n-1)/3}) \text{ if } x \text{ is on } \Gamma_s\,, \tag{4.22}$$

and thus the first term in the brackets in (4.15) vanishes to order $k^{-2(n-1)/3}$.

In order to make the second term in (4.15) vanish to a high order in k^{-1}, we shall require in Appendix B that

$$h(x, \alpha; k) = O(\alpha^{n-1}) \text{ for } x \text{ on } S, \tag{4.23}$$

which implies that

$$h(x, \alpha_l; k) = O(k^{-2(n-1)/3}) \text{ for } x \text{ on } S. \tag{4.24}$$

We note that our procedure up to this point differs slightly from the construction in [11], since there g and h are permitted to involve integral powers of $k^{-1/3}$, while we use only integral powers of k^{-1}. On the other hand, $\rho(x, \alpha_l)$ involves a sum of powers of $k^{-2/3}$, while the corresponding function in [11] involves only two terms. In Appendix B we construct $g(x, \alpha; k)$ and $h(x, \alpha; k)$ in such a way that (4.23) is satisfied. The construction involves the solution of ordinary differential equations along Γ_s, for example:

$$2\nabla\theta(x, 0) \cdot \nabla g_0(x, 0) + \Delta\theta(x, 0) g_0(x, 0) = 0. \tag{4.25}$$

Further details and interpretations of these equations are given in [11]. Now combining (4.15), (4.22) and (4.24), we see that

$$H^{(1)}(x, \alpha_l; k) = O(k^{-2(n-1)/3}) \text{ if } x \text{ is on } \Gamma_s. \tag{4.26}$$

In order to evaluate $\theta(x, \alpha_l)$, we note that, according to the construction of S_α, $\rho(x, \alpha)$ and $\theta(x, \alpha)$ (see (B.12)), we have

$$2\nabla\theta(x, 0) \cdot \nabla\theta_\alpha(x, 0) + (\nabla\rho(x, 0))^2 \rho_\alpha(x, 0) = 0 \text{ if } x \text{ is on } S. \tag{4.27}$$

In view of the fact that $\theta(x, 0) = s$ on S (see (1.11), (1.12)), $\nabla\theta(x, 0)$ is tangent to S. Thus, using (4.19), $\theta_\alpha(x, 0)$ can be determined from (4.27) if $(\nabla\rho(x, 0))^2$ is known. In the two dimensional case of a wave normally incident on a cylinder, the result was given by J. B. Keller [7] (see also [13], equation (1.59)):

$$(\nabla\rho(x, 0))^2 = \left(\frac{2}{a(s)}\right)^{2/3} \text{ for } x \text{ on } S, \qquad S \text{ a cylinder}, \tag{4.28}$$

where $a(s)$ is the radius of curvature of S at x. We conclude from (4.27) and (4.28) that in the two dimensional case

$$\theta_\alpha(x, 0) = \frac{1}{2}\int_{s_0}^{s}\left(\frac{2}{a(s')}\right)^{2/3} ds', \tag{4.29}$$

where s denotes arc length at x, and $s = s_0$ at the shadow boundary. An expression for $(\nabla\rho(x, 0))^2$ in terms of the relative curvatures of the surface ray on S and the tangential rays in space is given in [13], equation (2.64). An interpretation of this term and the corresponding expression for $\theta_\alpha(x, 0)$ is given in [11]. For present purposes, it is sufficient to observe that $\theta_\alpha(x, 0)$ is positive in the deep shadow. To first order, we have

$$\theta(x, \alpha_l) \sim \theta(x, 0) + \alpha_l \theta_\alpha(x, 0) , \tag{4.30}$$

and thus

$$|e^{ik\theta(x,\alpha_l)}| \sim \exp \{k^{1/3}\sqrt{\tfrac{3}{4}}\, q_l \theta_\alpha(x, 0)\} . \tag{4.31}$$

Since q_l is negative, $e^{ik\theta(x,\alpha_l)}$ is exponentially small in k^{-1} in the deep shadow. Combining (4.15), (4.26) and (4.31), we have

$$|u_l(x, k)| \sim \exp \{k^{1/3}\sqrt{\tfrac{3}{4}}\, q_l \theta_\alpha(x, 0)\} O(k^{-2(n-1)/3}) \text{ for } x \text{ on } \Gamma_s . \tag{4.32}$$

If x is in the deep shadow and away from S, we may use the asymptotic expansion (1.17) of $H^{(1)}(x, \alpha; k)$; thus we obtain

$$u_l(x, k) \sim 2ik\pi \frac{\tilde{H}^{(2)}(\alpha_l , k)}{\dfrac{\partial}{\partial \alpha} \tilde{H}^{(1)}(\alpha_l , k)} e^{ik\phi^+(x,\alpha_l)} z_+(x, \alpha_l ; k) . \tag{4.33}$$

We have, to first order,

$$\phi^+(x, \alpha_l) \sim \phi^+(x, 0) + \alpha_l \phi^+_\alpha(x, 0) . \tag{4.34}$$

In view of the definition (1.9) of $\phi^+(x, 0)$ and the fact that $\phi^+_\alpha(x, \alpha)$ is constant along rays corresponding to $\phi^+(x, \alpha)$, (4.34) becomes

$$\phi^+(x, \alpha_l) \sim s_+ + \alpha_l \theta_\alpha(x_+, 0) . \tag{4.35}$$

Here x is on a ray which is tangent to S at x_+, and $s_+ = \theta(x_+ , 0) = s(x_+)$. The result (4.33), (4.35) is identical with the diffracted wave predicted by J. B. Keller's geometrical theory of diffraction, [8]. Further details are given in [11].

In summary, we see that if the amplitudes J and $H^{(1)}$ agree in the deep shadow, then $u_i + u_s$ can be transformed into a sum of residues in the deep shadow. Each such residue will vanish on S to a high order in k^{-1} if $\rho(x, \alpha)$ and $h(x, \alpha)$ satisfy (4.18) and (4.23). The latter equations are the basis for the construction of S_α , θ, ρ, g and h in Appendix B. A further consequence of the construction of Appendix B is the identification of the terms of the residue sum with the diffracted or creeping waves predicted by the geometrical theory of diffraction.

5. Specification of the Amplitudes of J and $H^{(1)}$

In this section we shall construct functions g, h, $\hat{g}$ and $\hat{h}$ which satisfy all of the conditions imposed in Sections 2, 3 and 4. Thus the specification of the asymptotic solution (1.6), (1.7) will be complete. Finally, we shall justify the deformation of the contour of integration used in Section 4, by estimating the integrand on the displaced contour.

The following is a summary of the conditions which have been imposed on the functions appearing in (1.6), (1.7). In Section 2, in order that (1.6) might represent the incident field, we derived conditions which determine $\theta(x, \alpha)$ and $\rho(x, \alpha)$ completely (once S_α is known), but which specified $\hat{g}(x, \alpha; k)$ only on the curve C_α on S_α (see (2.11), (2.2) and (2.16)–(2.19)). Similarly, although $\tilde{\rho}(\alpha)$ was determined completely in Section 3, the amplitude $g(x, \alpha; k) + \sqrt{\rho(x, \alpha)}\, h(x, \alpha; k)$ was determined only where $\alpha = \tilde{\alpha}(x)$, i.e., along a certain curve R_α on S (see

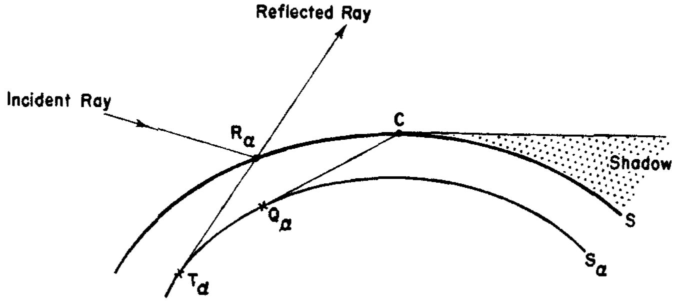

Figure 5.1

(3.21)–(3.23)). In Section 4, we required that $g(x, \alpha; k) = \hat{g}(x, \alpha; k)$ and $h(x, \alpha; k) = \hat{h}(x, \alpha; k)$ if x is in the shadow of S. We later required that $h(x, \alpha; k)$ vanish to order α^{n-1} on the shaded side of S, and that $\rho(x, \alpha) = -\alpha + O(\alpha^n)$ for x on S. The latter condition is used in Appendix B to construct S_α.

Now in order to specify $\hat{g}$ and $\hat{h}$, we stipulate that

$$\hat{h}(x, \alpha; k) = O(\alpha^{n-1}) \text{ if } x \text{ is on } S, \tag{5.1}$$

and we recall that $\hat{g}(x, \alpha; k)$ is prescribed in C_α by means of (2.11). The construction of Appendix B specifies $\hat{g}$ and $\hat{h}$ completely, using these two conditions.

In order to specify g and h, we set $\alpha < 0$ and we consider the curve Q_α on S_α where those outgoing rays (corresponding to $\phi^+(x, \alpha)$) which meet S at C (the shadow boundary) are tangent to S_α (see Figure 5.1.) Since the values of g and h on S to the right of C are determined by the values of g on S_α to the right of Q_α (see [13]), we will have $g(x, \alpha; k) = \hat{g}(x, \alpha; k)$ and $h(x, \alpha; k) = \hat{h}(x, \alpha; k)$ if x is in the shadow by requiring that

$$g(x, \alpha) = \hat{g}(x, \alpha) \text{ for } x \text{ on } S_\alpha, \qquad x \text{ to the right of } Q_\alpha. \tag{5.2}$$

On the other hand, $g(x, \alpha; k) + \sqrt{\rho(x, \alpha)}\, h(x, \alpha; k)$ is specified on the curve R_α on S, where $\tilde{\alpha}(x) = \alpha$. We note that R_α is to the left of C, since reflection takes place in the illuminated region. According to the transport equations satisfied by

g and h, the values of $g + \sqrt{\rho}\, h$ on R_α determine the values of g on T_α, T_α being the curve on S_α where the reflected rays at R_α are tangent to S_α. Since T_α is to the left of Q_α, there is no conflict with (5.2).

Now we define $g(x, \alpha; k)$ on S_α to the left of Q_α by a smooth interpolation and extrapolation of the values at T_α and Q_α. If α is bounded away from zero, then T_α is bounded away from Q_α, and there is no question that this can be done. In order to treat the case where α is near zero, it is shown in Appendix C that, at R_α,

$$g_j(x, \tilde{\alpha}) + \sqrt{\rho(x, \tilde{\alpha})}\, h_j(x, \tilde{\alpha}) = \hat{g}_j(x, \tilde{\alpha}) + \sqrt{\rho(x, \tilde{\alpha})}\, \hat{h}_j(x, \tilde{\alpha}) + O(\tilde{\alpha}^{n-2j}) . \tag{5.3}$$

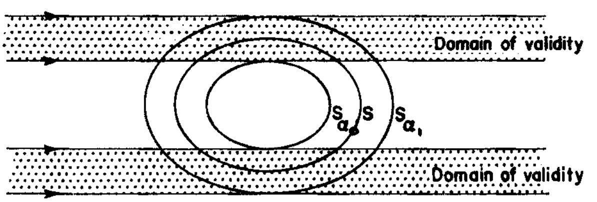

Figure 5.2

Here the subscript j denotes the coefficient of $(ik)^{-j}$ in the expansion of g or h. Consequently, for x on T_α,

$$g_j(x, \alpha) - \hat{g}_j(x, \alpha) = O(\alpha^{n-2j}) . \tag{5.4}$$

We conclude that for $n > 2j$, g can be defined on S_α so that (5.4) is valid everywhere. In case $\alpha \geqq 0$, we set $g(x, \alpha; k) = \hat{g}(x, \alpha; k)$ on S_α.

There is still a question about the interval of integration in the integrals (1.6), (1.7). The size of this interval is limited by certain geometrical restrictions which appear in the construction of S_α and in requirements that Δ (given by (2.14)), and $\tilde{\phi}_{\alpha\alpha}(x, \tilde{\alpha})$ (see (3.14)) be different from zero. There are numbers $\alpha_0 < 0$ and $\alpha_1 > 0$ such that all of our constructions are possible if $\alpha_0 \leqq \alpha \leqq \alpha_1$. Because we have taken no account of endpoint contributions in the asymptotic expansions of (1.6), (1.7), the integrands must vanish together with all of their derivatives at the ends of the interval. It is convenient for our purposes to multiply each of the amplitudes g, h, $\hat{g}$ and $\hat{h}$ as defined above by a factor $p(\alpha)$, given by

$$p(\alpha) = \exp\{-Bk^{-1/3}(\alpha - \alpha_0)^{-1}\} \exp\{-Bk^{-1/3}(\alpha_1 - \alpha)^{-1}\} , \tag{5.5}$$

where B is a positive number to be specified below. Thus our representations are valid in a cylindrical shell, whose size is independent of k, which is swept out by incident rays in a neighborhood of the shadow boundary, see Figure 5.2.

Now we shall justify the replacement of $u_i(x, k) + u_s(x, k)$ by a sum of residues where x is in the deep shadow. We deform the contour in (4.7) as shown in

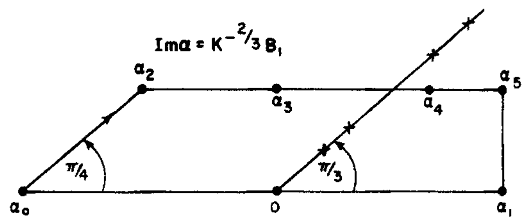

Figure 5.3

Figure 5.3, and estimate the integral on the deformed contour. Our final estimate is that the integral is less than u_L in (4.8) if the contour is displaced far enough into the upper half-plane.

Near $\alpha = \alpha_0$, we write $\alpha = \alpha_0 + k^{-2/3}\gamma e^{i\pi/4}$, where γ is real and positive. Between α_2 and α_5, we write $\alpha = \beta + ik^{-2/3}B_1$, where β and B_1 are real, $B_1 > 0$. From (5.5), we have

$$|p(\alpha)| \sim |\exp\{-Bk^{-1/3}(\alpha - \alpha_0)^{-1}\}| = \exp\left\{\frac{-Bk^{1/3}\gamma^{-1}}{\sqrt{2}}\right\} \tag{5.6}$$

for α between α_0 and α_2.

Hence $|p(\alpha)|$ has its maximum at α_2, and we have

$$|p(\alpha)| \leqq \exp\left\{-k^{1/3}\frac{B}{2B_1}\right\} \text{ for } \alpha \text{ between } \alpha_0 \text{ and } \alpha_2. \tag{5.7}$$

If x is in the deep shadow and $\mathscr{Re}\,\alpha \leqq \mathscr{Re}\,\alpha_3 < 0$, we may use the asymptotic expansions of $H^{(1)}$, $H^{(2)}$, $\tilde{H}^{(1)}$ and $\tilde{H}^{(2)}$:

$$\begin{aligned} &\frac{k}{2}\int_{\alpha_0}^{\alpha_3} e^{ik\theta}\left[H^{(2)}(x, \alpha; k) - H^{(1)}(x, \alpha; k)\frac{\tilde{H}^{(2)}(\alpha, k)}{\tilde{H}^{(1)}(\alpha, k)}\right] d\alpha \\ &\sim k\int_{\alpha_0}^{\alpha_3} [e^{ik\phi^-}z_- - e^{ik(\phi^+ + \psi^- - \psi^+)}z_+]\, d\alpha, \end{aligned} \tag{5.8}$$

where $z_\pm$ are given by (1.15) and $\psi^\pm$ are given by (3.3). We recall from Section 2 that ϕ_α^- is constant along rays corresponding to $\phi^-(x, \alpha)$. Thus if β is real,

$$\phi_\alpha^-(x, \beta) = \theta_\alpha(x_-(x, \beta), \beta); \tag{5.9}$$

$x_-(x, \beta)$ is the point on S_β where the ray passing through x (corresponding to ϕ^-) is tangent to S_β. Remembering that $\theta_\alpha(x, \beta) = 0$ if x is on C_β, we see that $\phi_\alpha^-(x, \beta) > B_2 > 0$ if $x_-(x, \beta)$ is in the shadow of S_β, in particular if x is in the shadow of S. Hence we obtain

$$\mathscr{Re}\,[ik\phi^-(x,\alpha)] \sim -k^{1/3}B_1\theta_\alpha(x_-(x,\beta),\beta) \leqq -k^{1/3}B_1B_2 \quad \text{if } \alpha \text{ is between } \alpha_2 \text{ and } \alpha_3\,. \tag{5.10}$$

In a similar fashion, we can estimate $\phi_\alpha^+ + \psi_\alpha^- - \psi_\alpha^+$: if $\beta \leqq 0$, we have

$$\phi_\alpha^+(x,\beta) + \psi_\alpha^-(\beta) - \psi_\alpha^+(\beta) = \phi_\alpha^+(x,\beta) - 2\rho_\alpha(x,\beta)\sqrt{\rho(x,\beta)} \geqq \phi_\alpha^+(x,\beta)\,, \tag{5.11}$$

since $\rho_\alpha(x,\beta) < 0$. Hence

$$\begin{aligned}\mathscr{Re}\,[ik(\phi^+(x,\alpha) + \phi^-(\alpha) - \phi^+(\alpha))] &\leqq -k^{1/3}B_1\theta_\alpha(x_+(x,\beta),\beta)\\ &\leqq -k^{1/3}B_1B_2 \text{ if } \alpha \text{ is between } \alpha_2 \text{ and } \alpha_3\,.\end{aligned} \tag{5.12}$$

Thus if α is between α_2 and α_3, the integrand in (5.8) has order $\exp\{-k^{1/3}B_1B_2\}$. Actually, a similar estimate is valid even if $\mathscr{Re}\,\alpha_3$ is near zero, provided that α_3 is not too close to a zero of $\tilde{H}^{(1)}(\alpha,k)$. Since $\phi_\alpha^-(x,\beta) > 0$ and $\phi_\alpha^+(x,\beta) + \psi_\alpha^-(\beta) - \psi_\alpha^+(\beta) > 0$, (5.7) implies that the integrand in (5.8) has order $\exp\{-k^{1/3}B/2B_1\}$ for α between α_0 and α_2. Now B_1 and B are still at our disposal. We may choose B_1 so large that $\exp\{-k^{1/3}B_1B_2\}$ is smaller than u_L in (4.8). Then B is chosen so that $B/2B_1 \geqq B_1B_2$.

In order to estimate the integrand for α between α_4 and α_1, we revert to the original expression (4.1) for $u_i + u_s$. For α between α_4 and α_1, we have $\tilde{\rho}(\alpha) \neq 0$. Thus $\tilde{J}$ and $\tilde{H}^{(1)}$ may be replaced by their asymptotic expansions, using the asymptotic expansion of the Airy function (see (1.18), (1.19) and [3]):

$$\tilde{J}(\alpha,k) \sim \exp\{-\tfrac{2}{3}k(-\tilde{\rho}(\alpha))^{3/2}\}Z_-(\alpha,k)\,, \tag{5.13}$$

$$\tilde{H}^{(1)}(\alpha,k) \sim 2\exp\{\tfrac{2}{3}k(-\tilde{\rho}(\alpha))^{3/2}\}Z_+(\alpha,k)\,. \tag{5.14}$$

The asymptotic expansion of $H^{(1)}(x,\alpha;k)$ might not be valid, since $\rho(x,\alpha)$ might be small, but we can use it as an estimate on $H^{(1)}(x,\alpha;k)$:

$$|H^{(1)}(x,\alpha;k)| \leqq B_3\exp\{\tfrac{2}{3}k\,|\rho(x,\alpha)|^{3/2}\}\,. \tag{5.15}$$

We conclude that for α between α_4 and α_1,

$$\left|H^{(1)}(x,\alpha;k)\,\frac{\tilde{J}(\alpha,k)}{\tilde{H}^{(1)}(\alpha,k)}\right| \leqq B_3\exp\{k\,\mathscr{Re}\,(\tfrac{2}{3}\,|\rho(x,\alpha)|^{3/2} - \tfrac{4}{3}(-\tilde{\rho}(\alpha))^{3/2}\}\,. \tag{5.16}$$

From Appendix C, we know that $\rho(x,\alpha) - \tilde{\rho}(\alpha) = O(\alpha^n)$ if x is on S, and hence the estimate (5.16) is exponentially small if x is on S. Since $-\rho(x,\alpha)$ decreases as x moves off S, the estimate is also valid if x is off S. In order to estimate the integral of the first term in (4.1), we may rewrite it as a double integral as in

Section 2. Since x is in the deep shadow, the stationary point of the integral corresponds to a negative value of α. We may apply a steepest descent procedure to the integral to obtain an exponentially small estimate.

Finally, for α between α_3 and α_4, a more delicate analysis is necessary in order to avoid the zeros of $\tilde{H}^{(1)}(\alpha, k)$ (see [15], Section IV). It is necessary to choose B_1 so that the contour passes midway between the zeros of $\tilde{H}^{(1)}(\alpha, k)$; the factor $e^{ik\theta(x,\alpha)}$ provides an appropriate exponential decay, since x is in the deep shadow.

6. The Boundary Condition in the Penumbra

In this section we shall verify that the boundary condition is satisfied in a neighborhood of the shadow boundary, i.e., that $u_i(x, k) + u_s(x, k) = O(k^{7/6-n/3})$ if x is on S. Since the solution was completely specified in the previous section, we have no free parameters or undetermined functions at our disposal. On the other hand, the analogy between the solution and the solution for the circular cylinder is especially close in the region of integration where α is small, which is crucial for the behavior of the solution in the penumbra. Our estimates are valid for points in the deep illumination or deep shadow, but they are more crude than those given in Sections 3 and 4. However, these estimates do show that there is no contribution from the interval $0 \leqq \alpha \leqq \alpha_1$ or its endpoints when evaluating u_s in the illuminated region.

By use of a partition of unity, we shall break the interval of integration into three overlapping intervals. In the first, where $|\alpha| \leqq 2Bk^{-1/3}$, $B > 0$, we shall use the results of Appendix C to show that

$$w = H^{(1)}(x, \alpha; k)\left[\frac{J(x, \alpha; k)}{H^{(1)}(x, \alpha; k)} - \frac{\tilde{J}(\alpha, k)}{\tilde{H}^{(1)}(\alpha, k)}\right] = O(k^{1/2-n/3}) . \tag{6.1}$$

Here and throughout this section, x is restricted to lie on S. It follows immediately that the corresponding integral has order $k^{7/6-n/3}$. In the interval where $-\alpha \geqq k^{-1/3}B$, the asymptotic expansions of $H^{(1)}$, J, $\tilde{H}^{(1)}$ and $\tilde{J}$ are valid, and the integrals for u_i and u_s may be evaluated by the method of stationary phase, as in Section 3. Since the incident and reflected waves cancel on S, the first n terms in the expansion of $u_i + u_s$ vanish. In the interval where $\alpha \geqq k^{-1/3}B$, we can again use the asymptotic expansions of $H^{(1)}$, J, $\tilde{H}^{(1)}$ and $\tilde{J}$. Since J and $\tilde{J}$ are exponentially small in k^{-1}, we conclude that the corresponding integral is exponentially small in k^{-1}.

We choose a function p_1 which is infinitely differentiable, such that $0 \leqq p_1(\gamma) \leqq 1$ for all γ,

$$p_1(\gamma) \equiv \begin{cases} 1 & \text{if} \quad |\gamma| \leqq B , \\ 0 & \text{if} \quad |\gamma| \geqq 2B . \end{cases} \tag{6.2}$$

Then we define $p_2(\gamma)$ and $p_3(\gamma)$ by means of

$$p_2(\gamma) = \begin{cases} 1 - p_1(\gamma) & \text{if} \quad \gamma < 0\,, \\ 0 & \text{if} \quad \gamma \geqq 0\,, \end{cases} \tag{6.3}$$

$$p_3(\gamma) = \begin{cases} 1 - p_1(\gamma) & \text{if} \quad \gamma > 0\,, \\ 0 & \text{if} \quad \gamma \leqq 0\,. \end{cases} \tag{6.4}$$

Thus we may write

$$u_i(x, k) + u_s(x, k) = I_1 + I_2 + I_3\,, \tag{6.5}$$

where

$$I_l(x, k) = k\int e^{ik\theta(x,\alpha)}\left[J(x, \alpha; k) - H^{(1)}(x, \alpha; k)\,\frac{\tilde{J}(\alpha, k)}{\tilde{H}^{(1)}(\alpha, k)}\right] p_l(k^{1/3}\alpha)\, d\alpha\,. \tag{6.6}$$

In order to estimate I_1 we estimate w, defined by (6.1). In view of the definitions of J, $H^{(1)}$, $\tilde{J}$ and $\tilde{H}^{(1)}$, we have

$$\begin{aligned} w = {} & k^{-1/3}[A(-e^{2\pi i/3}k^{2/3}\rho)g + ik^{-1/3}A'(-e^{-2\pi i/3}k^{2/3}\rho)h] \\ & \times \left[\frac{A(-k^{2/3}\rho)\hat{g} + ik^{-1/3}A'(-k^{2/3}\rho)\hat{h}}{A(-e^{2\pi i/3}k^{2/3}\rho)g + ik^{-1/3}A'(-k^{2/3}\rho)h} - \frac{A(-k^{2/3}\tilde{\rho})}{A(-e^{2\pi i/3}k^{2/3}\tilde{\rho})}\right]. \end{aligned} \tag{6.7}$$

From Section 5 and Appendix C, we have (see (5.1), (5.4) and (C.1))

$$\begin{aligned} h_j &= O(\alpha^{n-1-2j})\,, \\ \hat{h}_j &= O(\alpha^{n-1})\,, \\ g_j - \hat{g}_j &= O(\alpha^{n-2j})\,, \\ \rho - \tilde{\rho} &= O(\alpha^n)\,. \end{aligned} \tag{6.8}$$

Thus, in the support of $p_1(k^{1/3}\alpha)$, we have

$$\begin{aligned} h &= O(k^{-(n-1)/3})\,, \\ \hat{h} &= O(k^{-(n-1)/3})\,, \\ g - \hat{g} &= O(k^{-n/3})\,, \\ \rho - \tilde{\rho} &= O(k^{-n/3})\,. \end{aligned} \tag{6.9}$$

Since α is real, the denominators in (6.7) are bounded away from zero. If we regard w as a function of ρ, g, h and $\hat{h}$, an expansion around $\rho = \tilde{\rho}$, $g = \hat{g}$, $h = \hat{h} = 0$ shows that w has order $k^{1/2-n/3}$, as stated in (6.1), and hence I_1 has order $k^{7/6-n/3}$.

In the integral I_2 we have $-\alpha \geqq k^{-1/3}B$, and hence $k^{2/3}\rho(x, \alpha) \geqq B_1 k^{1/3}$. Thus J, $H^{(1)}$, $\tilde{J}$ and $\tilde{H}^{(1)}$ may be replaced by their asymptotic expansions. Assuming for the moment that I_2 may be evaluated by the method of stationary phase, the results of Section 3 show that, if $p_2(k^{1/3}\alpha) \equiv 1$ at the stationary points for u_i and u_s, then the first n terms of the expansion will cancel. If the stationary points lie in the region $k^{-1/3}B \leqq -\alpha \leqq 2k^{-1/3}B$, then cancellation will be affected by the factor $p_2(k^{2/3}\alpha)$. Denoting the stationary points for u_i and u_s by $\hat{\alpha}$ and $\tilde{\alpha}$ as in Sections 2 and 3, Appendix C (C.11), shows that $\hat{\alpha} - \tilde{\alpha} = \sqrt{\rho(x, \tilde{\alpha})}\, O(\tilde{\alpha}^n) = O(k^{-1/6-n/3})$. Consequently we have

$$p_2(k^{1/3}\tilde{\alpha}) - p_2(k^{1/3}\hat{\alpha}) = O(k^{1/6-n/3}) , \tag{6.10}$$

with similar estimates for the derivatives of p_2 ; thus the stationary phase contributions will cancel to the same order. If the stationary points lie outside the interval of integration, i.e., $0 \leqq -\alpha \leqq k^{-1/3}B$, then we shall see presently that I_2 is smaller than any power of k^{-1}. On the shaded side of S, we may use (4.7), and no stationary points appear.

We turn to the justification of the method of stationary phase. Using the asymptotic expansions of J, $H^{(1)}$, $\tilde{J}$ and $\tilde{H}^{(1)}$, we have (see (3.4) and (1.15))

$$I_2 \sim k\int\Big[e^{ik\phi^-(x,\alpha)}\tilde{z}_- + e^{ik\phi^+}\tilde{z}_+ - e^{ik\phi^+}z_+ - e^{ik(\phi^+ + \psi^- - \psi^+)}z_+\frac{\tilde{z}_-}{\tilde{z}_+}\Big]\, p_2(k^{1/3}\alpha)\, d\alpha . \tag{6.11}$$

The term involving $\phi^-(x, \alpha)$ is typical. We replace α as variable of integration by β, defined by

$$\beta^2 = \rho(x, \alpha) , \tag{6.12}$$

and we introduce $s(x, \beta)$ by means of

$$s(x, \beta) = \frac{\theta(x, \alpha) - \theta(x, 0)}{\rho(x, \alpha)} . \tag{6.13}$$

Then we have

$$\phi^-(x, \alpha) = \theta(x, \alpha) - \tfrac{2}{3}(\rho(x, \alpha))^{3/2} = \theta(x, 0) + \beta^2 s - \tfrac{2}{3}\beta^3 . \tag{6.14}$$

The results in [14] assure us that the method of stationary phase may be applied to the first term in (6.11) as long as $k^{1/3}s$ is large. Thus if $s \geqq \frac{1}{2}Bk^{-1/6}$, the remainder after n terms in the stationary phase evaluation of $u_i + u_s$ will have order $k^{1-n/2}$. If $s \leqq \frac{1}{2}Bk^{-1/6}$, then the stationary point for ϕ^- lies outside the support of

p_2. Using the results of [14], we conclude that in the latter case the first term in I_2 is smaller than any power of k^{-1}. Similar estimates hold for the remaining terms of I_2, involving ϕ^+ and $\phi^+ + \psi^- - \psi^+$.

In the integral I_3, we may again use the asymptotic expansions of J, $H^{(1)}$, $\tilde{J}$ and $\tilde{H}^{(1)}$. As in (5.13)–(5.16), we conclude that I_3 is exponentially small in k^{-1}.

Appendix A. Diffraction by a Circular Cylinder

In the case of diffraction of a plane wave by a circular cylinder, the exact solution can be given explicitly. Since the situation is simpler in that case, our procedure is more transparent. The methods and results of this appendix are drawn from S. I. Rubinow and J. B. Keller [16] and H. M. Nussenzveig [15].

We consider two-dimensional space: $x = (x_1, x_2)$. The function $u(x, k)$ is required to satisfy (1.1)–(1.3), with S replaced by a cylinder of radius a and $u_i(x, k) = e^{ikx_1}$.

It is convenient to introduce polar coordinates $x_1 = r\cos\theta$, $x_2 = r\sin\theta$. The polar angle θ is related to, but not the same as, the function $\theta(x, \alpha)$ used in other parts of this work. By means of the Poisson sum formula (representing $e^{ikr\cos\theta}$ by means of a Fourier transform with respect to θ), we obtain

$$u_i = e^{ikr\cos\theta} = k\sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty} e^{ik\beta(-\theta+\pi/2+2\pi m)} J_{k\beta}(kr)\, d\beta\,. \tag{A.1}$$

First we shall discuss the asymptotic expansion of $J_{k\beta}(kr)$, and then we shall show that, at any given point, all but one of the terms of the sum (A.1) are negligible. For $r > |\beta|$, $J_{k\beta}(kr)$ may be represented by means of its Debye expansion:

$$e^{ik\beta(-\theta+\pi/2)} J_{k\beta}(kr) \sim e^{ik\phi^+} z_+ + e^{ik\phi^-} z_-\,. \tag{A.2}$$

Here we have set

$$\phi^\pm(x, \beta) = \beta\left(\frac{\pi}{2} - \theta_\pm\right) \pm t_\pm \qquad \text{if} \qquad \beta > 0\,, \tag{A.3}$$

$$\phi^\pm(x, \beta) = \beta\left(-\frac{\pi}{2} - \theta_\pm\right) \pm t_\pm \qquad \text{if} \qquad \beta < 0\,, \tag{A.4}$$

where $\theta_\pm$ and $t_\pm$ are the angles and lengths shown in Figure A.1.[2] Hence, as $|x| \to \infty$, $\phi^+ \to \infty$ and $\phi^- \to -\infty$, i.e., $e^{ik\phi^+} z_+$ is outgoing and $e^{ik\phi^-} z_-$ is incoming. The functions $\phi^\pm$ also satisfy the eikonal equation (1.10) identically in β, and hence by differentiating (1.10) we see that $\phi^\pm_\beta(x, \beta)$ is constant along the rays which correspond to $\phi^\pm$.

[2] The radius of the circle in Figure A.1 is β.

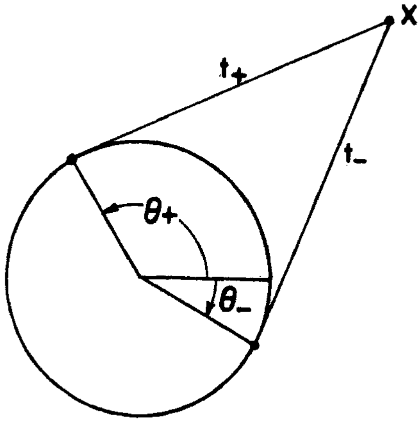

Figure A.1

Now we apply the method of stationary phase to (A.1), after replacing the integrand by (A.2). The condition of stationary phase is

$$\phi_{\tilde\beta}^{\pm}(x, \tilde\beta) = -2\pi m \,. \tag{A.5}$$

If $|x| = \tilde\beta$, then from (A.3) we must have $\theta_\pm = 2\pi m + \frac{1}{2}\pi$. Hence, the phase is stationary along the horizontal ray which is tangent to the circle $|x| = \tilde\beta$ at the top. There is another stationary point with $\beta < 0$, corresponding to the horizontal ray which is tangent at the bottom. The result of the stationary phase evaluation, of course, is the incident wave; compare the treatment in Section 2. If we think of polar coordinates as giving a many-sheeted covering of the plane, then each term in (A.1) is significant on only one sheet; the summation over m is required to make the result single-valued. We are interested in values of θ near $\frac{1}{2}\pi$, and hence we may neglect all terms but one in the sum (A.1).

We are now ready to represent the scattered field. Since u_s is outgoing, u_s should be a superposition of Hankel functions of the first kind. The condition that $u_i + u_s = 0$ for $|x| = a$ then implies that

$$u_s(x, k) = -k \int_{-\infty}^{\infty} H_{k\beta}^{(1)}(kr) e^{ik\beta(\pi/2-\theta)} \frac{J_{k\beta}(ka)}{H_{k\beta}^{(1)}(ka)} \, d\beta \,. \tag{A.6}$$

Only the integration where $|\beta| \leqq a$ is significant, since $J_{k\beta}(ka)$ is exponentially small if $|\beta| > a$, and the ratio of the Hankel functions is bounded.

The following discussion is the prototype for Section 3. In the illuminated region, we can replace the Bessel and Hankel functions by their Debye expansions. Then (A.6) becomes

$$\begin{aligned} u_s(x, k) \sim & -k \int e^{ik\phi^+(x,\beta)} z_+(x, \beta) \, d\beta \\ & -k \int e^{ik(\phi^+(x,\beta)+\psi^- - \psi^+)} z_+(x, \beta, k) \frac{z_-(a, \beta; k)}{z_+(a, \beta; k)} \, d\beta \,, \end{aligned} \tag{A.7}$$

where $\psi^+(a, \beta) = -\psi^-(a, \beta) = t_+(x)$ evaluated at $|x| = a$. The first integral in (A.7) can be evaluated by stationary phase; we obtain

$$\phi_\beta^+(x, \tilde{\beta}) = 0 \qquad \text{at the stationary point} . \tag{A.8}$$

This condition is only satisfied if x is on the right half of a horizontal ray which is tangent to a circle of radius $\tilde{\beta}$. Since we may restrict ourselves to $|\beta| \leqq a$, the first integral has a stationary point only in the shadow.

In the second integral in (A.7), the condition of stationary phase is

$$\phi_\beta^+(x, \tilde{\beta}) - \psi_\beta^+(a, \tilde{\beta}) + \psi_\beta^-(a, \tilde{\beta}) = 0 . \tag{A.9}$$

We first consider the case where $|x| = a$. In view of the definitions of $\phi^\pm$ and $\psi^\pm$, (A.9) becomes

$$\phi_\beta^-(x, \tilde{\beta}) = 0 \qquad \text{for} \qquad |x| = a , \tag{A.10}$$

i.e., x is on the incident ray which is tangent to the circle of radius $\tilde{\beta}$. At the stationary point (not necessarily on $|x| = a$) the total phase is

$$\tilde{\phi}(x) = \phi^+(x, \tilde{\beta}) + \psi^-(a, \tilde{\beta}) - \psi^+(a, \tilde{\beta}) , \tag{A.11}$$

and hence

$$\nabla\tilde{\phi}(x) = \nabla\phi^+(x, \tilde{\beta}) . \tag{A.12}$$

Symmetry shows that if $|x| = a$, $\nabla\phi^+$ and $\nabla\phi^-$ make equal angles with the normal at x. Thus if dx is tangent to the circle of radius a and ϕ_i denotes the incident phase, we have

$$dx \cdot \nabla\tilde{\phi}(x) = dx \cdot \nabla\phi^+(x, \tilde{\beta}) = dx \cdot \nabla\phi^-(x, \tilde{\beta}) = dx \cdot \nabla\phi_i(x) . \tag{A.13}$$

We conclude that $\nabla\tilde{\phi}$ has the direction of the reflected ray at x, if $|x| = a$. Since $\tilde{\beta}$ is constant if $\nabla\tilde{\phi}$ is constant, we conclude that $\tilde{\phi}(x)$ is the reflected phase, and our evaluation of (A.6) yields the reflected wave in the illuminated region. However, the procedure breaks down at the shadow boundary.

In order to discuss the deep shadow region (the analogue of Section 4), we combine (A.1) and (A.6) to obtain

$$u_i(x, k) + u_s(x, k) = k\int_{-\infty}^{\infty} e^{ik\beta(\pi/2-\theta)}\left[J_{k\beta}(kr) - H_{k\beta}^{(1)}(kr)\,\frac{J_{k\beta}(ka)}{H_{k\beta}^{(1)}(ka)}\right] d\beta . \tag{A.14}$$

Since

$$J_{k\beta}(kr) = \tfrac{1}{2}H_{k\beta}^{(1)}(kr) + \tfrac{1}{2}H_{k\beta}^{(2)}(kr) , \tag{A.15}$$

we may rewrite (A.14) as

$$u_i(x, k) + u_s(x, k) = \tfrac{1}{2}k\int_{-\infty}^{\infty} e^{ik\beta(\pi/2-\theta)}\left[H_{k\beta}^{(2)}(kr) - H_{k\beta}^{(1)}(kr)\,\frac{H_{k\beta}^{(2)}(ka)}{H_{k\beta}^{(1)}(ka)}\right] d\beta . \tag{A.16}$$

If the contour of integration is deformed into the upper half-plane, the integral is replaced by a sum of residues at the points β_l such that

$$H^{(1)}_{k\beta_l}(ka) = 0 \,. \tag{A.17}$$

Each such residue will have the form

$$u_l(x, k) = -i\pi k \frac{H^{(2)}_{k\beta_l}(ka)}{\frac{\partial}{\partial \beta} H^{(1)}_{k\beta_l}(ka)} e^{ik\beta_l\theta} H^{(1)}_{k\beta_l}(kr) \,. \tag{A.18}$$

The asymptotic expansions of β_l and $u_l(x, k)$ may be obtained by use of uniform asymptotic expansions of $H^{(1)}$ and $H^{(2)}$ in terms of Airy functions. The interpretation of $u_l(x, k)$ as a creeping wave is given in Section 4.

Appendix B. Construction of ρ, θ, $\hat{g}$ and $\hat{h}$

In this appendix, we shall construct $\rho(x, \alpha)$, $\theta(x, \alpha)$, $\hat{g}(x, \alpha; k)$ and $\hat{h}(x, \alpha; k)$ in such a way as to satisfy the following conditions:

$$\rho(\nabla\rho)^2 + (\nabla\theta)^2 = 1 \,, \tag{B.1}$$

$$2\nabla\rho \cdot \nabla\theta = 0 \,, \tag{B.2}$$

$$\rho(x, \alpha) = -\alpha + O(\alpha^n) \text{ for } x \text{ on } S \,, \tag{B.3}$$

$$2\nabla\theta \cdot \nabla\hat{g} + \Delta\theta\hat{g} + 2\rho\nabla\rho \cdot \nabla\hat{h} + \rho\Delta\rho\hat{h} + (\nabla\rho)^2\hat{h} + \frac{1}{k}\Delta\hat{g} = 0 \,, \tag{B.4}$$

$$2\nabla\rho \cdot \nabla\hat{g} + \Delta\rho\hat{g} + 2\nabla\theta \cdot \nabla\hat{h} + \Delta\theta\hat{h} + \frac{1}{k}\Delta\hat{h} = 0 \,, \tag{B.5}$$

$$\hat{h}(x, \alpha; k) = O(\alpha^{n-1}) \text{ for } x \text{ on } S \,. \tag{B.6}$$

It is easily verified that (B.1), (B.2) are equivalent to (1.10) and (1.12), (1.13) (the eikonal equation for $\phi^{\pm}$), and (B.4), (B.5) are equivalent to the transport equations for the amplitudes $z_{\pm}$ associated with $\phi^{\pm}$. It is further shown in [13] that (B.1), (B.2) and (B.4), (B.5) imply that $e^{ik\theta(x,\alpha)}J(x, \alpha; k)$ is an asymptotic solution of the reduced wave equation. Conditions (B.3) and (B.6) are used in Sections 4, 5 and 6. Our solution will permit $\theta(x, \alpha)$ and $\hat{g}(x, \alpha; k)$ to be specified on S_α as in Section 2.

We shall first give a formal solution of (B.1)–(B.3). This formal solution will be used to construct the surfaces S_α. Then an actual solution of (B.1)–(B.3) can be obtained from the theory of [13]. The construction of $\hat{g}$ and $\hat{h}$ proceeds in an analogous fashion.

We look for a (formal) solution of (B.1), (B.2) of the form

$$\hat{\rho}(x, \alpha) = \sum_{j=0}^{\infty} \hat{\rho}_j(x) \frac{\alpha^j}{j!}, \tag{B.7}$$

$$\hat{\theta}(x, \alpha) = \sum_{j=0}^{\infty} \hat{\theta}_j(x) \frac{\alpha^j}{j!}, \tag{B.8}$$

where (B.3) is replaced by

$$\hat{\rho}(x, \alpha) = -\alpha \text{ if } x \text{ is on } S. \tag{B.9}$$

Setting $\alpha = 0$ in (B.1), (B.2), we observe that $\hat{\rho}_0$ and $\hat{\theta}_0$ satisfy (B.1), (B.2). In view of (2.11), (2.12), we require that $\hat{\theta}_0(x) = \phi_i(x)$ and $\nabla\hat{\theta}_0(x) = \nabla\phi_i(x)$ if x is on C (the shadow boundary). According to [13], Sections 1 and 2, $\theta_0(x)$ can be determined on S, and then $\hat{\rho}_0$ and $\hat{\theta}_0$ can be constructed in the exterior of S by integrating along rays which are tangent to S. The resulting functions have analytic continuations inside S.

By substituting (B.7), (B.8) into (B.1), (B.2) and equating each coefficient of α to zero, we see that $\hat{\rho}_1$ and $\hat{\theta}_1$ must satisfy

$$\hat{\rho}_1(\nabla\rho_0)^2 + 2\hat{\rho}_0\nabla\hat{\rho}_0 \cdot \nabla\hat{\rho}_1 + 2\nabla\hat{\theta}_0 \cdot \nabla\hat{\theta}_1 = 0, \tag{B.10}$$

$$\nabla\hat{\rho}_0 \cdot \nabla\hat{\theta}_1 + \nabla\hat{\rho}_1 \cdot \nabla\hat{\theta}_0 = 0, \tag{B.11}$$

with similar linear equations for $\hat{\rho}_j$ and $\hat{\theta}_j$. On S, (B.9) implies that $\hat{\rho}_0 = 0$ and $\hat{\rho}_1 = -1$. Thus, on S, (B.10) becomes

$$2\nabla\hat{\theta}_0 \cdot \nabla\hat{\theta}_1 = (\nabla\hat{\rho}_0)^2 \text{ for } x \text{ on } S. \tag{B.12}$$

The fact that S has Gauss curvature bounded away from zero implies that $(\nabla\hat{\rho}_0)^2 \neq 0$ (see [13], Section 2 or [11]) and hence the tangential derivative of $\hat{\theta}_1$ is different from zero. In view of (2.9), we set $\hat{\theta}_1 = 0$ on C. Thus $\hat{\theta}_1$ can be determined on S from (B.12). After multiplying (B.11) by $\pm 2\sqrt{\rho_0}$ and adding (B.10), we obtain

$$2(\nabla\hat{\theta}_0 \pm \sqrt{\hat{\rho}_0}\,\nabla\hat{\rho}_0) \cdot (\nabla\hat{\theta}_1 \pm \sqrt{\hat{\rho}_0}\,\nabla\hat{\rho}_1) = 0. \tag{B.13}$$

Hence $\hat{\theta}_1 \pm \sqrt{\rho_0}\,\hat{\rho}_1$ is constant along rays which correspond to $\phi^{\pm}(x, 0)$, and $\hat{\theta}_1$ and $\hat{\rho}_1$ can be determined from (B.13). The functions $\hat{\theta}_j$ and $\hat{\rho}_j$ can be constructed in a similar fashion. The appropriate values of $\hat{\theta}_j$ on C can be computed by differentiating (2.9).

We cannot define ρ and θ by means of (B.7), (B.8), since the corresponding series will probably converge only in exceptional cases. This remark is due to

J. Moser, who observed an analogy with certain series which are known to diverge. We can circumvent this difficulty by truncating the series: we set

$$\hat{\rho}^{(n)}(x, \alpha) = \sum_{j=0}^{n-1} \hat{\rho}_j(x) \frac{\alpha^j}{j!}, \tag{B.14}$$

$$\hat{\theta}^{(n)}(x, \alpha) = \sum_{j=0}^{n-1} \hat{\theta}_j(x) \frac{\alpha^j}{j!}. \tag{B.15}$$

Now we define S_α as the locus where $\hat{\rho}^{(n)}(x, \alpha) = 0$. We obtain $\rho(x, \alpha)$ and $\theta(x, \alpha)$ which satisfy (B.1), (B.2) by prescribing $\rho(x, \alpha) = 0$ on S_α and using (2.11), (2.12). The existence of $\rho(x, \alpha)$ and $\theta(x, \alpha)$ is assured by the results in [13].

We shall now verify that

$$\theta(x, \alpha) - \hat{\theta}^{(n)}(x, \alpha) = O(\alpha^n), \tag{B.16}$$

$$\rho(x, \alpha) - \hat{\rho}^{(n)}(x, \alpha) = O(\alpha^n). \tag{B.17}$$

From our construction we immediately have

$$\hat{\rho}_0(x) = \rho(x, 0), \tag{B.18}$$

$$\hat{\theta}_0(x) = \theta(x, 0). \tag{B.19}$$

From (2.9), we also have

$$\hat{\theta}_1(x) = \theta_\alpha(x, 0) \qquad \text{on} \qquad C. \tag{B.20}$$

Now let $z(\alpha)$ be a point on S_α. By definition we have

$$\hat{\rho}^{(n)}(z(\alpha), \alpha) = 0, \tag{B.21}$$

$$\rho(z(\alpha), \alpha) = 0. \tag{B.22}$$

Differentiating (B.21), (B.22) with respect to α and setting $\alpha = 0$, we obtain

$$\hat{\rho}_1 + \nabla \hat{\rho}_0 \cdot z_\alpha(0) = 0, \tag{B.23}$$

$$\rho_\alpha(x, 0) + \nabla \rho(x, 0) \cdot z_\alpha(0) = 0. \tag{B.24}$$

Thus, since $\hat{\rho}_0(x) = \rho(x, 0)$, we have

$$\rho_\alpha(x, 0) = \hat{\rho}_1(x) = -1 \text{ if } x \text{ is on } S. \tag{B.25}$$

In a similar fashion, by differentiating (B.1), (B.2) and comparing with (B.10), (B.11), we see that

$$\hat{\rho}_1(x) = \rho_\alpha(x, 0) , \tag{B.26}$$

$$\hat{\theta}_1(x) = \theta_\alpha(x, 0) . \tag{B.27}$$

By a similar procedure, the remaining terms in $\hat{\rho}^{(n)}$ and $\hat{\theta}^{(n)}$ can be identified with appropriate derivatives of ρ and θ at $\alpha = 0$.

We turn to the construction of $\hat{g}$ and $\hat{h}$ which satisfy (B.4)–(B.7). The appropriate surface on which to specify $\hat{g}$ and $\hat{h}$ is S_α ; however, condition (B.6) refers to S. This difficulty (like the analogous difficulty in the construction of ρ and θ) can be circumvented by a formal expansion of $\hat{g}$ and $\hat{h}$ in powers of α and k, i.e.,

$$\hat{g}(x, \alpha; k) = \sum_{l=0}^{\infty} \sum_{j=0}^{\infty} \hat{g}_{lj}(x)(ik)^{-l} \frac{\alpha^j}{j!} , \tag{B.28}$$

$$\hat{h}(x, \alpha; k) = \sum_{l=0}^{\infty} \sum_{j=0}^{\infty} \hat{h}_{lj}(x)(ik)^{-l} \frac{\alpha^j}{j!} . \tag{B.29}$$

Setting $\alpha = 0$ and inserting (B.28), (B.29) into (B.4), (B.5), we see that (B.4), (B.5) are satisfied by $\hat{g}(x, 0; k)$ and $\hat{h}(x, 0; k)$, and (B.6) becomes

$$\hat{h}(x, 0; k) = 0 \text{ for } x \text{ on } S . \tag{B.30}$$

In view of (B.30) and the fact that $\rho(x, 0) = 0$ on S, (B.4) becomes

$$2\nabla\theta(x, 0) \cdot \nabla\hat{g}(x, 0; k) + \Delta\theta(x, 0)\hat{g}(x, 0; k) + \frac{1}{k} \Delta\hat{g}(x, 0; k) = 0 . \tag{B.31}$$

This is the surface transport equation on S; it determines $\hat{g}(x, 0; k)$ as a formal power series in k^{-1} if $\hat{g}(x, 0; k)$ is prescribed on S. The equation for $\hat{g}_{00}$ is especially noteworthy: from (B.31) we obtain

$$2\nabla\theta(x, 0) \cdot \nabla\hat{g}_{00}(x) + \Delta\theta(x, 0)\hat{g}_{00}(x) = 0 . \tag{B.32}$$

Now $\hat{g}(x, 0; k)$ and $\hat{h}(x, 0; k)$ can be obtained by integrating along the rays which correspond to $\phi^\pm(x, 0)$ (see [13]).

By inserting (B.28), (B.29) into (B.4), (B.5) and equating the coefficient of each power of α to zero, we obtain equations analogous to (B.4), (B.5) for derivatives of $\hat{g}$ and $\hat{h}$ with respect to α at $\alpha = 0$, which also can be solved by integrating along rays. This formal solution can be used to provide the appropriate values for $\hat{g}(x, \alpha; k)$ on S_α. Finally, application of the procedure of [13] enables us to construct $\hat{g}$ and $\hat{h}$ from (B.4), (B.5), with the assurance that (B.6) will be satisfied.

Appendix C. The Connection Between ρ and $\tilde{\rho}$, g and $\hat{g}$

The functions $\rho(x, \alpha)$ and $\hat{g}(x, \alpha; k)$ were defined in connection with the representation of the incident field in Section 2, and the functions $\tilde{\rho}(\alpha)$ and $g(x, \alpha; k)$ were defined in connection with the representation of the reflected field in Section 3. In order that the boundary condition be satisfied in the penumbra, it is necessary that

$$\rho(x, \alpha) - \tilde{\rho}(\alpha) = O(\alpha^n) \text{ for } x \text{ on } S\,. \tag{C.1}$$

It follows from (B.3) and (C.1) that

$$\tilde{\rho}(\alpha) = -\alpha + O(\alpha^n)\,, \tag{C.2}$$

and it is an easy consequence of (B.3) and (C.2) that g and $\hat{g}$ satisfy

$$\begin{aligned} &g_j(x, \tilde{\alpha}) + \sqrt{\rho(x, \tilde{\alpha})}\, h_j(x, \tilde{\alpha}) \\ &\qquad = \hat{g}_j(x, \tilde{\alpha}) + \sqrt{\rho(x, \tilde{\alpha})}\, \hat{h}_j(x, \tilde{\alpha}) + O(\tilde{\alpha}^{n-2j}) \text{ for } x \text{ on } \Gamma_i\,. \end{aligned} \tag{C.3}$$

We shall derive these facts together with some related information in this appendix.

From the definition (1.12) of $\rho(x, \alpha)$, we have

$$\tfrac{4}{3}(\rho(x, \alpha))^{3/2} = \phi^+(x, \alpha) - \phi^-(x, \alpha)\,. \tag{C.4}$$

Combining (3.10) and (C.4) and observing from (2.8) that $\phi_i(x) = \phi^-(x, \hat{\alpha}(x))$, we have

$$\tfrac{4}{3}(\tilde{\rho}(\tilde{\alpha}))^{3/2} = \tfrac{4}{3}(\rho(x, \tilde{\alpha}))^{3/2} + \phi^-(x, \tilde{\alpha}) - \phi^-(x, \hat{\alpha})\,, \tag{C.5}$$

or from (1.12), (1.13),

$$\begin{aligned} \tfrac{4}{3}(\tilde{\rho}(\tilde{\alpha}))^{3/2} &= \tfrac{4}{3}(\rho(x, \tilde{\alpha}))^{3/2} + \theta(x, \tilde{\alpha}) \\ &\qquad - \theta(x, \hat{\alpha}) - \tfrac{2}{3}[(\rho(x, \tilde{\alpha}))^{3/2} - (\rho(x, \hat{\alpha}))^{3/2}]\,. \end{aligned} \tag{C.6}$$

In order to estimate $\tilde{\rho}(\tilde{\alpha}) - \rho(x, \tilde{\alpha})$ from (C.6), we first estimate $\tilde{\alpha}(x) - \hat{\alpha}(x)$. From (2.10) and (3.9), we have

$$\begin{aligned} dx \cdot \nabla\phi^+(x, \tilde{\alpha}) &= dx \cdot \nabla\phi_i(x) \\ &= dx \cdot \nabla\phi^-(x, \hat{\alpha}) \text{ if } x \text{ is on } \Gamma_i\,, \qquad dx \text{ tangent to } \Gamma_i\,. \end{aligned} \tag{C.7}$$

If we express ϕ^+ and ϕ^- in terms of θ and ρ, (C.7) becomes

$$dx \cdot [\nabla\theta(x, \tilde{\alpha}) - \nabla\theta(x, \hat{\alpha}) + \sqrt{\rho(x, \tilde{\alpha})}\, \nabla\rho(x, \tilde{\alpha}) + \sqrt{\rho(x, \hat{\alpha})}\, \nabla\rho(x, \hat{\alpha})] = 0\,. \tag{C.8}$$

It follows from differentiation of (B.3) that

$$dx \cdot \nabla\rho(x, \alpha) = O(\alpha^n) \text{ if } x \text{ is on } \Gamma_i, \qquad dx \text{ tangent to } \Gamma_i. \tag{C.9}$$

Substituting (C.9) into (C.8), we obtain

$$dx \cdot [\nabla\theta(x, \tilde{\alpha}) - \nabla\theta(x, \hat{\alpha})] = \sqrt{\rho(x, \tilde{\alpha})}\, O(\tilde{\alpha}^n) + \sqrt{\rho(x, \hat{\alpha})}\, O(\hat{\alpha}^n), \tag{C.10}$$

or

$$dx \cdot \nabla\theta_\alpha(x, \tilde{\alpha})(\tilde{\alpha} - \hat{\alpha}) + O(\tilde{\alpha} - \hat{\alpha})^2 = \sqrt{\rho(x, \tilde{\alpha})}\, O(\tilde{\alpha}^n) + \sqrt{\rho(x, \hat{\alpha})}\, O(\hat{\alpha}^n). \tag{C.11}$$

In view of (B.13), we have $dx \cdot \nabla\theta_\alpha \neq 0$ if dx has the direction of $\nabla\theta(x, 0)$. Thus we conclude from (C.11) that

$$\tilde{\alpha} - \hat{\alpha} = \sqrt{\rho(x, \tilde{\alpha})}\, O(\tilde{\alpha}^n). \tag{C.12}$$

In view of (C.12), (C.6) becomes

$$\tfrac{4}{3}(\tilde{\rho}(\tilde{\alpha}))^{3/2} = \tfrac{4}{3}(\rho(x, \tilde{\alpha}))^{3/2} + \sqrt{\rho(x, \tilde{\alpha})}\, O(\tilde{\alpha}^n). \tag{C.13}$$

Our main result (C.1) follows from (C.13).

The following remarks are useful for Section 3. From the definition of ϕ^+ we have

$$dx \cdot \nabla\phi_\alpha^+(x, \tilde{\alpha}) = dx \cdot \nabla\theta(x, \tilde{\alpha}) + \frac{1}{2}\frac{\rho_\alpha}{\sqrt{\rho}} \nabla\rho \cdot dx + \sqrt{\rho}\, \nabla\rho_\alpha \cdot dx \tag{C.14}$$

$$\text{if } x \text{ is on } \Gamma_i, \qquad dx \text{ tangent to } \Gamma_i.$$

In view of (B.3) and (C.9), we have

$$dx \cdot \nabla\phi_\alpha^+(x, \tilde{\alpha}) = dx \cdot \nabla\theta_\alpha(x, \tilde{\alpha}) + \frac{1}{\sqrt{\rho}} O(\tilde{\alpha}^n). \tag{C.15}$$

Thus if dx has the direction of $\nabla\theta(x, 0)$, we conclude from (B.12) that

$$\nabla\theta(x, 0) \cdot \nabla\phi_\alpha^+(x, \tilde{\alpha}) \neq 0 \text{ for } x \text{ on } \Gamma_i. \tag{C.16}$$

Differentiating (C.2) in the direction of $\nabla\theta(x, 0)$, we also obtain

$$\nabla\theta(x, 0) \cdot \nabla\tilde{\alpha}(x) \neq 0, \tag{C.17}$$

since differentiation of (2.4), (2.5) implies that

$$\nabla\theta(x, 0) \cdot \nabla\hat{\alpha}(x) \neq 0. \tag{C.18}$$

Now we examine the relationship between $g(x, \tilde{\alpha}; k)$ and $\hat{g}(x, \hat{\alpha}; k)$. We obtained $\hat{g}(x, \hat{\alpha}; k) + \hat{\xi}\hat{h}(x, \hat{\alpha}; k)$ by applying the method of stationary phase to the double integral (2.1). The same result is obtained on Γ_i by using the asymptotic expansion of $J(x, \alpha; k)$ and applying stationary phase to (1.6). The coefficients $\hat{g}_j(x, \hat{\alpha}) - \sqrt{\rho(x, \hat{\alpha})}\, \hat{h}_j(x, \hat{\alpha})$ thus are obtained in terms of the derivatives up to order $2j$ of

$$\hat{\phi}(x, \hat{\alpha}) = \theta(x, \hat{\alpha}) - \tfrac{2}{3}(\rho(x, \hat{\alpha}))^{3/2} . \tag{C.19}$$

On the other hand, in Section 3, $g_j(x, \tilde{\alpha}) + \sqrt{\rho(x, \tilde{\alpha})}\, h_j(x, \tilde{\alpha})$ was determined in terms of the derivatives up to order $2j$ of

$$\tilde{\phi}(x, \tilde{\alpha}) = \theta(x, \tilde{\alpha}) + \tfrac{2}{3}(\rho(x, \tilde{\alpha}))^{3/2} - \tfrac{4}{3}(\tilde{\rho}(\tilde{\alpha}))^{3/2} . \tag{C.20}$$

In view of (C.12), (C.13),

$$\hat{\phi}(x, \hat{\alpha}) - \tilde{\phi}(x, \tilde{\alpha}) = \sqrt{\rho(x, \tilde{\alpha})}\, O(\tilde{\alpha}^n) . \tag{C.21}$$

Since the results of the stationary phase evaluations are both equal to $e^{ik\phi_i(x)} z_i(x, k)$ on Γ_i, we conclude that

$$\begin{aligned} \hat{g}_j(x, \hat{\alpha}) - \sqrt{\rho(x, \hat{\alpha})}\, \hat{h}_j(x, \hat{\alpha}) &= g_j(x, \tilde{\alpha}) + \sqrt{\rho(x, \tilde{\alpha})}\, h_j(x, \tilde{\alpha}) \\ &\quad + \sqrt{\rho(x, \tilde{\alpha})}\, O(\tilde{\alpha}^{n-2j}) . \end{aligned} \tag{C.22}$$

It follows from (5.1) and (C.12) that

$$\begin{aligned} \hat{g}_j(x, \tilde{\alpha}) + \sqrt{\rho(x, \tilde{\alpha})}\, \hat{h}_j(x, \tilde{\alpha}) &= g_j(x, \tilde{\alpha}) + \sqrt{\rho(x, \tilde{\alpha})}\, h_j(x, \tilde{\alpha}) \\ &\quad + \sqrt{\rho(x, \tilde{\alpha})}\, O(\tilde{\alpha}^{n-2j}) . \end{aligned} \tag{C.23}$$

Bibliography

[1] Babich, V. M., *On the short wavelength asymptotics of the Green's function for the exterior of a bounded, convex region*, Doklady Akad. Nauk SSSR, Vol. 146, No. 3, 1962. (In Russian.)

[2] Buslaev, V. S., *On formulas of short wave asymptotics in the problem of diffraction by convex bodies*, Akad. Nauk SSSR Trudy Math. Inst. Steklov Vol. 73, Math., Fiz. 2, 1964, pp. 14–117. (In Russian.)

[3] Erdélyi, A., *Asymptotic Expansions*, Dover Publ., New York, 1965.

[4] Erdélyi, A., Magnus, W., Oberhettinger, F., and Tricomi, F. G., *Tables of Integral Transforms*, Vol. 2, McGraw-Hill, New York, 1954.

[5] Fok, V. I., *Diffraction, refraction and reflection of radio waves*, Air Force Cambridge Research Center, translations, Bedford, Mass., 1957.

[6] Grimshaw, R., *High-frequency scattering by finite convex regions*, Comm. Pure Appl. Math., Vol. 19, 1966, pp. 167–198.

[7] Keller, J. B., *A geometrical theory of diffraction*, *Calculus of Variations and its Applications*, Proc. Symp. Applied Math., Vol. 8, pp. 27–52, McGraw-Hill, New York, 1958.

[8] Keller, J. B., *Geometrical theory of diffraction*, J. Optical Soc. Amer., Vol. 52, 1962, pp. 116–130.

[9] Kravtsov, Yu. A., *A modification of the geometrical optics method*, Radiofizika, Vol. 7, 1964, pp. 664–673. (In Russian.)
[10] Kravtsov, Yu. A., *Asymptotic solution of Maxwell's equations near a caustic*, Radiofizika, Vol. 7, 1964, pp. 1049–1056. (In Russian.)
[11] Lewis, R. M., Bleistein, N., and Ludwig, D., *Uniform asymptotic theory of creeping waves*, Comm. Pure Appl. Math., to appear in Vol. 20, No. 2, 1967.
[12] Logan, N. A., and Yee, T. S., *A mathematical model for diffraction by convex surfaces, Electromagnetic Waves*, edit. R. E. Langer, Univ. of Wisconsin Press, 1962.
[13] Ludwig, D., *Uniform asymptotic expansions at a caustic*, Comm. Pure Appl. Math., Vol. 19, 1966, pp. 215–250.
[14] Ludwig, D., *An extension of the validity of the stationary phase formula*, to appear in SIAM J. on Appl. Math.
[15] Nussenzveig, H. M., *High frequency scattering by an impenetrable sphere*, Annals of Physics, Vol. 34, 1965, pp. 23–95.
[16] Rubinow, S. I., and Keller, J. B., *Shift of the shadow boundary and scattering cross section of an opaque object*, J. Appl. Phys., Vol. 32, 1961, pp. 814–820.
[17] Zauderer, E., *An Integral representation approach to diffraction by a smooth object*, to appear in J. Math. Anal. Appl., 1967.
[18] Brown, W. P., Jr., *On the asymptotic behavior of electromagnetic fields scattered from convex cylinders near grazing incidence*, J. Math. Anal., Vol. 15, 1966, pp. 355–385.

Received April, 1966.