Acoustic modal analysis

F.Augusztinovicz
KULeuven, Belgium

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1. Introduction

The phenomena, related to the existence of acoustic modes, were already known in the ancient world and our ancestors, though instinctively, have even exploited some of the acoustic effects [1]. The first treatments of scientific character of the field date back to the 19th century [2,3] while the basics of the modal theory of room acoustics were developed in the first half of this century [4 - 7]. Nevertheless, a revival of the acoustic modal theory and its experimental aspects seems to be worthwhile for a couple of reasons.

In the first place, the ever increasing demand for lower noise levels and personal comfort necessitates to attack also those problems, for which the traditional armoury of noise control engineering is no longer sufficient. The booming noise in cars or the propeller noise in small aircrafts are typical examples of a low frequency noise problem, where the consideration of the single acoustic modes is inevitable.

In the second place, a wide range of modern computational, experimental and noise control techniques are based on or closely related to the acoustic modes. We can cite here both the Finite Element and the Boundary Element method, active noise and/or vibration control of sound field in closed spaces and others.

A further and very important reason of the renewed interest in acoustical modal analysis is that the rapid development in the experimental techniques of structural dynamics has just recently enabled us to render the acoustic modal analysis method from a pure theoretical calculation procedure, burdened with serious application limitations, to an experimental engineering routine. This transition is however not without dangers, since the analyst can obtain misleading results if the structural methods are used for acoustic applications without due foresight.

The aim of this course notes is to summarise the basic notions of the acoustical modal theory and the inter-relations thereof, to shed light on the existence and limitations of the analogy between the modal behaviour of mechanical and acoustic systems and to give some hints for those who are interested in the practical details in the experimental modal analysis in acoustics and vibro-acoustics.

2. Physics and mathematics of modes in one-dimensional acoustic systems

Acoustic waves are longitudinal vibrations, taking place in gaseous fluids. Below we survey some important notions and descriptors of these vibrations, first in plausible physical terms for simple, one-dimensional cases. One-dimensional acoustic models bear some practical importance, too, but their most important merit for us lies in the fact that all the important notions and relationships can more readily be demonstrated. The physical visualisation will be supported by a more rigorous mathematical derivation and the treatment is extended to account for damping and three-dimensional wave propagation. Eventually, the theory will be exemplified by the results of some simple model experiments.

2.1. Formation of modes in undamped case

The formation of the modes in acoustical systems can most easily be demonstrated by means of a simple one-dimensional device: a rigid, circular, straight tube terminated at one end by a closely fitted piston and rigidly closed at the other end as shown in Fig. 1. [8]. Assume that the piston is forced to a periodical, say sinusoidal motion. Let the lateral dimension of the tube be negligible with respect to the wavelength in the gaseous fluid filling the tube. Then the acoustical field is independent of the coordinates different from x, and we face a pure one-dimensional problem. For the sake of simplicity first we neglect any source of damping.

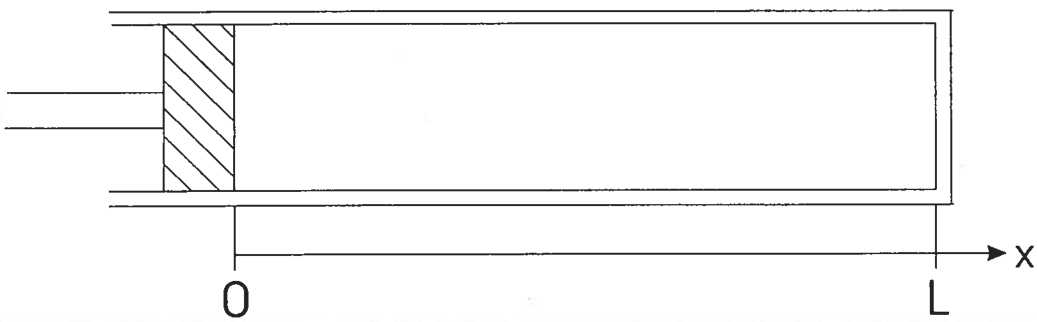

Fig.1. Sketch of a simple one-dimensional waveguide

Assume now that the oscillating piston generates a wave, propagating in $+x$ direction:

$$p_+(x,t) = p_1 e^{j(\omega t - kx)} \tag{1a}$$

and, due to the Euler-relation:

$$v_+(x,t) = -\frac{1}{\rho}\int \nabla p\, dt = \frac{p_1}{\rho c} e^{j(\omega t - kx)} \tag{1b}$$

The wave propagates until it reaches the end of the tube where it is reflected, resulting in a wave propagating in the *-x* direction:

$$p_-(x,t) = p_2 e^{j(\omega t + kx)} \tag{2a}$$

and

$$v_-(x,t) = -\frac{p_2}{\rho c} e^{j(\omega t + kx)} \tag{2b}$$

The conditions of this reflection are determined by the acoustic impedance of the termination. According to the assumptions made above the end of the tube is perfectly rigid, thereby the particle velocity at $x = L$ must vanish, irrespective of time:

$$v_{+(x=L)} + v_{-(x=L)} = \frac{(p_1 - p_2)}{\rho c} e^{jkL} = 0$$

that is, the pressure amplitude of the impinging and the reflected waves must be equal ($p_1 = = p_2 = p_0$). Using this result in Eqs. (1) and (2) and comparing them one obtains that in case of a rigid termination the pressure of the reflected wave remains undisturbed while the particle velocity is phase reversed.

An important consequence of this total reflection is that in an arbitrary point at $0 < x < L$ the pressure will be

$$p(x) = 2\,p_0 \cos kx\; e^{j\omega t} \tag{3a}$$

$$v(x) = -j\frac{2\,p_0}{\rho c} \sin kx\; e^{j\omega t} \tag{3b}$$

Eqs. (3a) and (3b) represent a special type of wave, the phase of which no longer depends on the spatial variable *x*. The pressure and the velocity reach their maxima and minima everywhere along the tube in the same time; what changes is only the value of these maxima. These waves are therefore called *standing waves*.

An important feature of these standing waves is that the pressure and the velocity are always 90° out of phase. Consequently, the real intensity of the standing wave is everywhere zero, meaning that there is no net energy transport along the tube.

Assuming now that the piston is also perfectly rigid, the same reflection phenomenon occurs at the left termination of the tube once again. As we have seen, the rigid termination does not alter the pressure but reverses the phase of the particle velocity. Therefore, after two reflections and a complete round-trip both the phase and the particle velocity are identical as if the wave would have travelled straightforward along a distance of $2L$. Moreover, if this distance is equal to an integer multiple of the wavelength λ, too, than the reflected wave reinforces the pressure of the new primary wave, radiated by the piston. The constructive interference of the reflected wave with the earlier radiated ones takes place after every round-trip, resulting in a very high, in principle infinite, amplitude. This phenomenon is referred to as *acoustical resonance*. The frequency at which the effect takes place is called *resonance frequency;* its value can be calculated from the condition mentioned above and turns out to be

$$\omega_n = \frac{nc\pi}{L}, \qquad n = 1, 2, \ldots, \infty \tag{4}$$

In reality, due to various damping mechanisms the amplitude always remains bounded. The detailed mathematical analysis of a weakly damped, one-dimensional waveguide, terminated by very high acoustic impedances shows [8] that the asymptotic (steady state) solution is described by the equations

$$p_d(x) = P \frac{\cos kx \, \sin \omega t}{\sin kL} \tag{5a}$$

and

$$v_d(x) = \frac{P}{\rho c} \frac{\sin kx \, \cos \omega t}{\sin kL} \tag{5b}$$

Obviously, these equations continue to represent a standing wave, with pressure maxima and velocity minima at both ends of the tube. Resonance is also manifested because p and v become singular when $kL = n\pi$; this condition is identical with Eq. (4).

Returning for a while to the undamped case again, imagine that after some round-trips of the wave the piston is suddenly stopped. In lack of energy absorption the waves remain prevalent in the tube. If the excitation frequency of the piston was originally equal to one of the resonance frequencies, the wave motion in the tube remains undisturbed and no longer requires a source for its maintenance. Such natural constant-frequency disturbances are referred to as *modes, normal modes* or *eigenmodes* and occur only for certain discrete frequencies, usually termed *natural, modal* or *eigenfrequencies.* The spatial variation along the tube is characteristic of the eigenvibration in the system, too; this function is usually called *mode shape.*

It is very important to note here that the normal modes are intrinsic characteristics of the system itself, and they are independent from any external forcing effect. On the contrary, the resonance effect can only be brought about by means of some periodic excitation, corresponding to some modal frequency of the system. (These two notions are closely related and hence often mixed up in the everyday usage.) As to the standing waves, they can be observed both in normal modes and in forced vibrations, and even in cases where no modes can exist and, therefore, no resonance can take place; so e.g. in front of a single rigid wall. In other words, standing waves are necessary but not sufficient conditions of the existence of normal modes.

Strictly speaking, this last remark is valid only for undamped cases. As we will see later, in case of energy dissipation the mode shape becomes complex and does not correspond to a true standing wave any longer. Nevertheless, these waves are usually also referred to as standing waves.

2.2. Analysis of a waveguide, terminated by general acoustic impedances

2.2.1. Mathematical analysis

We will now repeat the analysis of a one-dimensional acoustic waveguide on the basis of a general acoustic formulation: the acoustic wave equation [21].

As noted above, consider a finite circular tube of length L and of diameter d so that $d<<L$. Let the air in the tube be at rest, with uniform density and temperature. Then the wave motion in the tube can be described by the one-dimensional acoustic wave equation

$$\frac{\partial^2 p}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \tag{6a}$$

and similarly for the particle velocity

$$\frac{\partial^2 v}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 v}{\partial t^2} \tag{6b}$$

Seeking for a solution to Eq.(6a) in the form

$$p(x,t) = p_x(x)\, p_t(t)$$

the spatial and the temporal variables can be separated:

$$\frac{d^2 p_x(x)}{dx^2} + k^2\, p_x(x) = 0 \tag{7a}$$

$$\frac{1}{c^2}\frac{d^2 p_t(t)}{dt^2} + k^2\, p_t(t) = 0 \tag{7b}$$

Eq.(7a) is the familiar homogeneous Helmholtz-equation (here in one-dimensional form), describing the spatial variation of the pressure along the tube while Eq.(7b) can be used to determine the temporal variation. . The solution $p_x(x)$ has to satisfy both the equation and the boundary conditions at the end of the tube. We face here a classical eigenvalue problem: one has to determine, for which k values does a non-trivial solution exist and what is the sound field associated with these eigenvalues.

The solution is assumed to be in the usual complex form:

$$p(x,t) = p_0\, e^{j\omega t}\left(a e^{-jkx} + b e^{jkx}\right) \equiv \psi(x) e^{j\omega t} \tag{8a}$$

The particle velocity can again be obtained from the Euler-relation:

$$v(x,t) = -\frac{1}{j\omega\rho}\frac{\partial p}{\partial x} = \frac{p_0}{\rho c}\, e^{j\omega t}\left(a e^{-jkx} - b e^{jkx}\right) \tag{8b}$$

To obtain the modes of the investigated system, Eq.(7a) has to be solved with appropriate boundary conditions at the ends of the tube. Assume that the tube is closed at one end ($x = 0$) by a specific acoustic impedance Z_1:

$$\left.\frac{p_x(x)}{v_x(x)}\right|_{x=0} = -Z_1 \tag{9}$$

and similarly at $x = L$:

$$\left.\frac{p_x(x)}{v_x(x)}\right|_{x=L} = Z_2 \tag{10}$$

(Note that the negative sign in Eq.(9) is caused by the fact that the outward normal of the enclosing surface is opposite to the direction of the x-axis.) Substituting Eqs. (8a) and (8b) in Eqs. (9) and (10) we get the characteristic equation of the system:

$$\frac{Z_1+\rho c}{Z_1-\rho c}\,\frac{Z_2+\rho c}{Z_2-\rho c}\, e^{j2kL} = 1 \tag{11}$$

Fortunately, it can be solved in closed form by using the complex logarithm function and then one obtains for the nth modal frequency a complex value:

$$\omega = \omega_n + j\delta$$

where the real value of ω ,

$$\omega_n \equiv \frac{nc\pi}{L} - \frac{c\phi\{\zeta_{12}\}}{2L} \tag{12}$$

represents the frequency of the eigenvibration while the imaginary value stands for the damping, caused by the energy losses in the system:

$$\delta \equiv \frac{c \ln|\zeta_{12}|}{2L} \tag{13}$$

and

$$\zeta_{12} \equiv \frac{Z_1+\rho c}{Z_1-\rho c}\,\frac{Z_2+\rho c}{Z_2-\rho c} \tag{14}$$

Calculating now the wavenumber

$$k = \frac{\omega}{c} = k_n + j\kappa$$

and substituting back to the homogeneous Helmholtz-equation, the mode shape can eventually be calculated:

$$\psi(x) = \frac{\cosh \kappa x (Z_1 \cos k_n x + j\rho c \sin k_n x)}{Z_1 - \rho c} - \frac{\sinh \kappa x (\rho c \cos k_n x + j Z_1 \sin k_n x)}{Z_1 - \rho c} \tag{15}$$

2.2.2. *Qualitative discussion of some special cases*

To discuss the behaviour of our system quantitatively, it is instructive to calculate the complex modal frequencies and the mode shapes for various simple parameter combinations.

First assume that the closing impedances are purely imaginary: $Z = jX$. The modal frequency then becomes purely real, but different from what we have obtained for the tube closed at both ends (as expressed in Eq.(4)):

$$\omega_n = \frac{nc\pi}{L} - \frac{c \tan^{-1}(\rho c / X)}{L} \tag{16}$$

Depending on the sign of the closing reactances, the impedances at the end of the tube make the tube seemingly longer or shorter. Depicting the mode shape, Eq.(15), in the Nyquist plot, one gets a straight line having a slope which depends on the ratio of *X/rc* as depicted in Fig. 2. These kind of modes are usually referred to in structural dynamics as *real modes*, but in our case it seems to be more correct to use the notation *collinear modes* since the mode shape functions are actually complex functions.

If the terminating (identical) impedances are purely real and greater than the specific acoustic impedance of the fluid *(rc)*, the damped modal frequency remains unchanged with respect to the rigid-rigid termination but one gets finite damping:

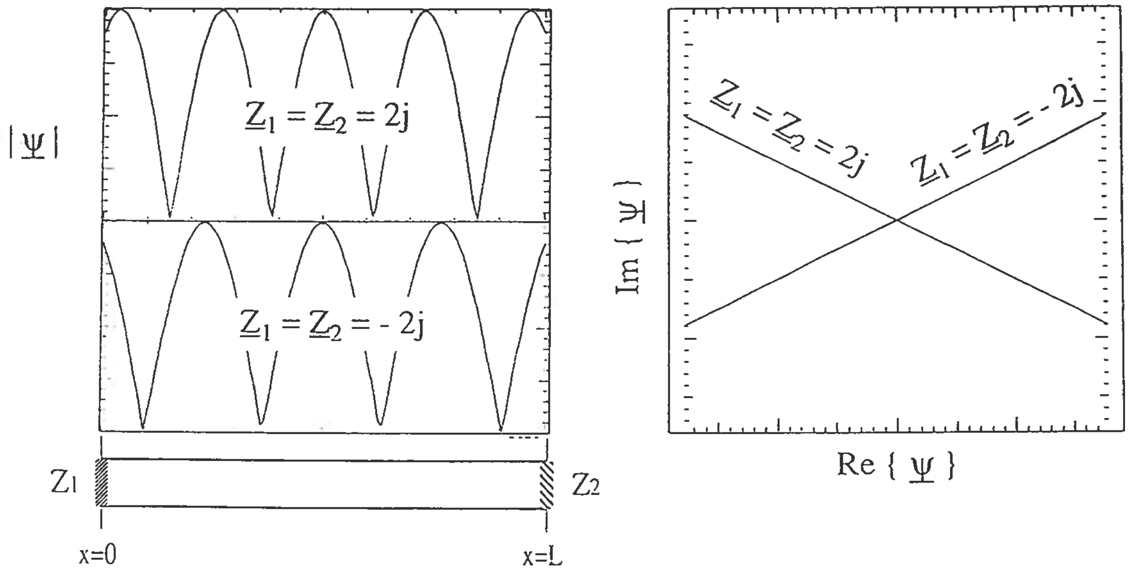

Fig.2. Calculated mode shapes of a one-dimensional waveguide, terminated by pure imaginary impedances

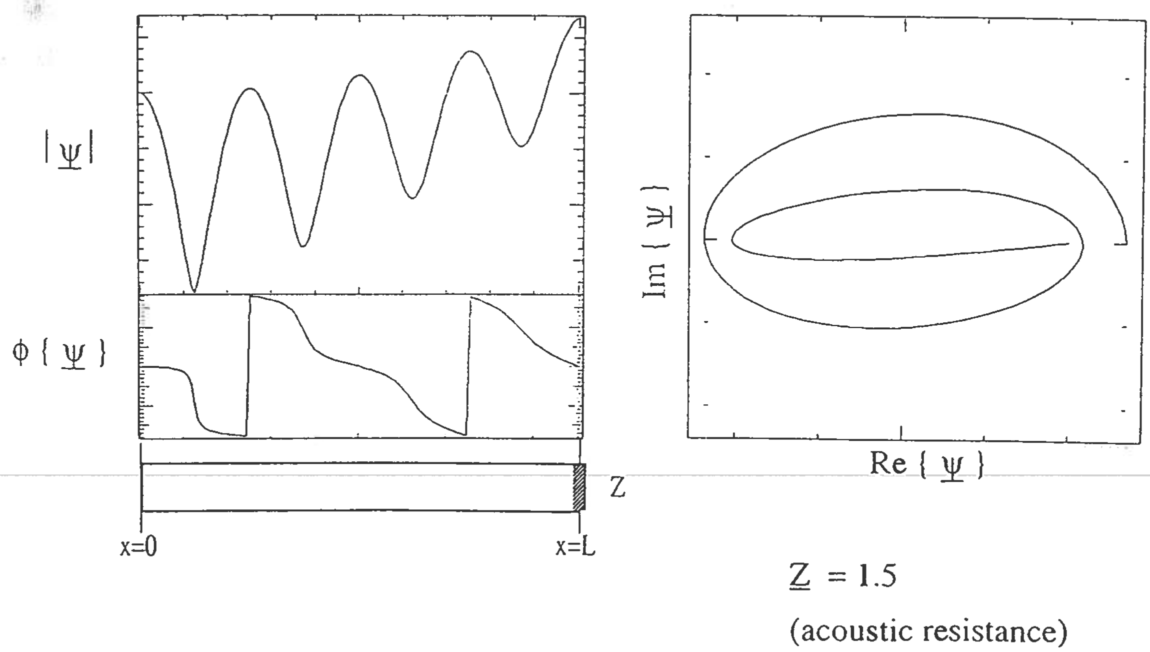

Fig.3. Calculated mode shape of a one-dimensional waveguide, terminated by pure real impedance

$$\delta = \ln\left[(R+\rho c)/(R-\rho c)\right]/L \tag{17}$$

The mode shape is accordingly no longer collinear but becomes a truly complex one as shown in Fig. 3.

In the general case both effects take place: the mode will be damped and the modal frequency is shifted, and the mode shapes will be complex, too. The Nyquist plots of some typical mode shapes are shown in Fig.4.

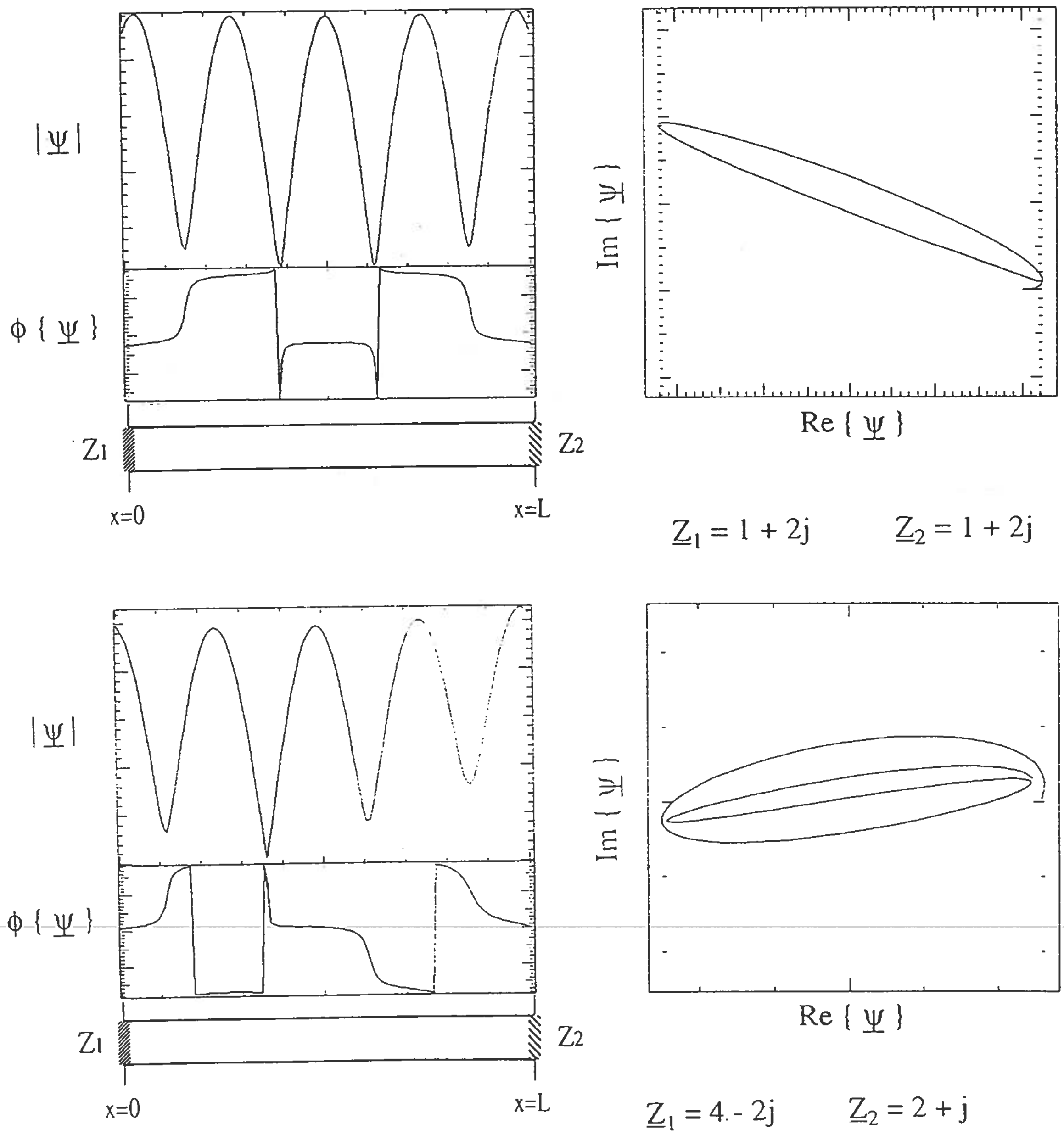

Fig.4. Calculated mode shapes of one-dimensional waveguides, terminated by various general impedances

2.2.3. *Verification experiments*

The aim of our experiments was to conduct an acoustical modal analysis test for the verification of the calculations discussed above. The apparatus used for the experiments was a standard acoustic impedance tube of Type Brüel & Kjær 4002. This instrument is normally used for measurements of the specific acoustic impedance and absorption coefficient of circular cut samples of acoustic materials. The design and the dimensions of the tube ensure that only one-dimensional wave propagation can exist in the tube within the specified frequency range.Unlike in the course of the standard procedure, the excitation of the tube was ensured by means of a random noise generator and the excitation signal was parallel fed to the input of an experimental modal analysis system. The sound field was sampled by means of a small electret condenser microphone Type AKG CK-67/3, traversed along the axis of the tube by means of the original microphone carriage system. The first modal analysis test was carried out without any sound absorbing material. Then a 5 cm thick polyurethane foam disc was inserted in the sample holder and the experiment was repeated. As anticipated, the first experiment resulted in a number of lightly damped eigenfrequencies of the tube. It is interesting to see, however, that the frequencies are not strictly harmonic; the slight deviations are thought to be caused by the finite acoustic impedance of the driving loudspeaker. The residuals of the experimental acoustic modal analysis test, which are proportional to the mode shape values, are shown in Fig. 5a, depicted both as a spatial function along the tube and in a Nyquist plot. In agreement with our findings above, the locus is nearly collinear and the mode shape corresponds to a clear standing wave. In case of the absorber at the end of the tube, see Fig. 5b, the locus of the residuals is truly complex.

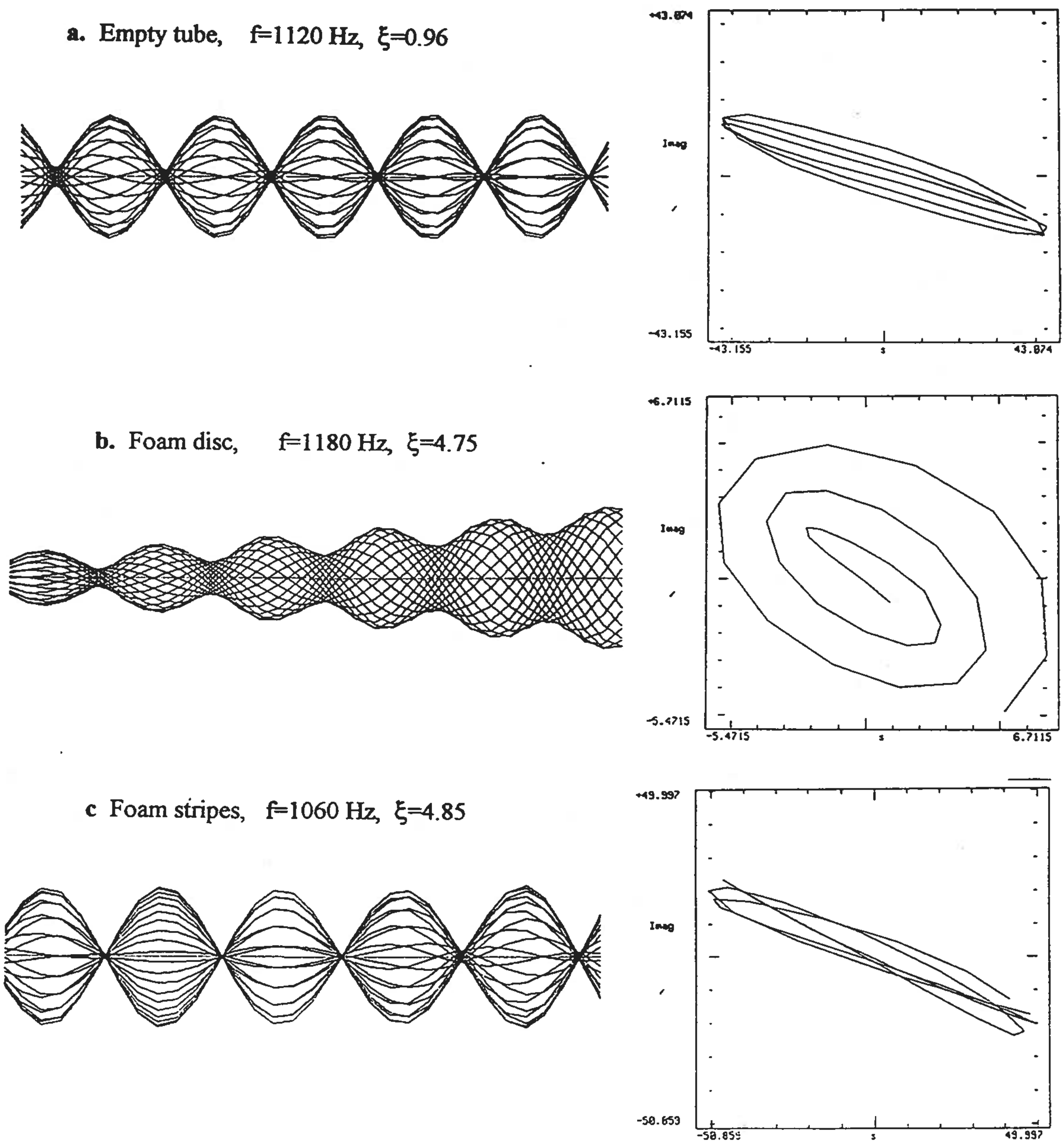

Fig.5. Measured mode shapes and locii of residuals of a one-dimensional waveguide.

a. Tube rigidly terminated,

b. tube terminated by a 50mm thick foam absorber,

c. tube provided with absorber stripes along the whole tube

2.3. Analysis with distributed damping

The above analysis represents a rather special case, with non-rigid boundary conditions at the ends of a one-dimensional vibrating system only. It is more realistic to assume that the damping is distributed along the length of the tube. The mathematical treatment of this problem is feasible but too complex to repeat here in full details; the reader is rather referred to the relevant literature [9,10]. Nevertheless, it is instructive to show the results of a simple experiment: the modal analysis of the tube discussed in paragraph 2.2.3 with stripes of sound absorbing material along the whole tube. The measuring system and the analysis technique is just the same as it was before. As one can see in Fig.5c., the modal frequency is again different from the case with rigid termination but the mode shape corresponds very well to the undamped situation, in spite of the fact that the modal damping of the test is higher than it was with the sound absorbing termination.
This observation can be explained by means of the analogy between mechanical and acoustical systems, to be evolved later on in paragraph 5.1. As it is known from structural dynamics, it is not the damping itself which is responsible for the complexity of the mode shape but rather the unproportional distribution of it. (Of course, if damping is present in a system, the modal frequencies must always be complex.) Unproportional damping is an inherent characteristics of the overwhelming majority of acoustical systems, since the viscous or thermal conductivity losses of acoustic fluids are usually negligible with respect to damping caused by the dissipative boundaries of the fluid in the relevant frequency range of modal analysis. (In order to avoid confusion with notions of structural dinamics, we will call unproportional damping of acoustic systems as *local damping*). As a consequence, in most of the cases one encounters complex modes, the interpretation of which can cause various problems.

3. Modes and forced waves in three-dimensional systems

3.1. Modes in undamped systems

If we want to extend our analysis for a general, three-dimensional, bounded space, the one-dimensional wave equation, Eq.(6a) has to be written in the more general form of

$$\nabla^2 p = \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} \tag{18}$$

The space-dependent Helmholtz-equation then becomes

$$\left(\nabla^2 + k^2\right) p = 0 \tag{19}$$

This equation is not easy to solve in closed form, unless one uses a coordinate system in which the variables are separable and the boundary conditions are simple enough. A classical, simple, yet important case is a rectangular room as shown in Fig. 6. Assume that the boundaries are perfectly rigid; then we have the boundary conditions

$v_x = 0$ at $x = 0$ and $x = L_x$, (20a)

$v_y = 0$ at $y = 0$ and $y = L_y$, (20b)

$v_z = 0$ at $z = 0$ and $z = L_z$, (20c)

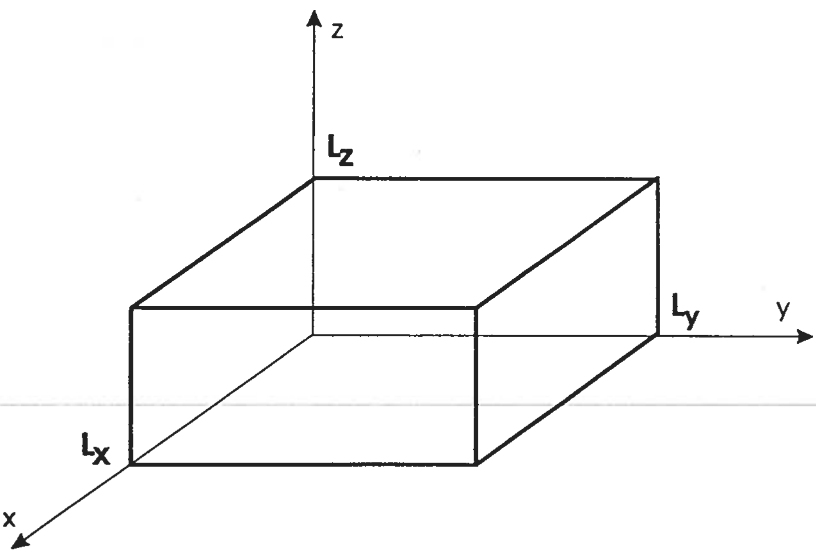

Fig.6. Sketch of a simple three-dimensional room

Since the individual boundary conditions depend on only one of the coordinates, they can be fulfilled independently. Let us compose the solution by the product of three functions, each dependent only on *x*, *y* and *z*, respectively:

$$p(x,y,z) = p_x(x)\, p_y(y)\, p_z(z) \tag{21a}$$

It is rather plausible that all these terms differ from each other in the independent variables only and the spatial variation in one of the directions should be the same as in the one-dimensional case [7]. Thus we obtain:

$$p(x,y,z) = cos\,\frac{n_x \pi x}{L_x}\; cos\,\frac{n_y \pi y}{L_y}\; cos\,\frac{n_z \pi z}{L_z} \tag{21b}$$

where the numbers n_x, n_y and n_z are non-negative integers. The corresponding eigenfrequency is then :

$$\omega_n = \sqrt{\left(\frac{n_x \pi c}{L_x}\right)^2 + \left(\frac{n_y \pi c}{L_y}\right)^2 + \left(\frac{n_z \pi c}{L_z}\right)^2}\,. \tag{22}$$

If only one of the numbers are different from zero, we regain the one-dimensional solution expressed by Eq.(4); these modes are called *axial modes*. In cases when two values are finite, we speak of *tangential modes* and in the general case of *oblique modes*.

The eigenfrequencies of a three dimensional rectangular room can be visualised in several ways. For axial or tangential modes the neutral or nodal lines can be shown as depicted in Fig. 7a. For tangential modes the wavefronts can also be drawn, see Fig. 7b. The true three-dimensional "modal model" can be depicted as distorted wireframes, by colors etc. Nevertheless, the schematic representation of the modal frequencies by means of a three-dimensional lattice does not only represent the modes in a very concise way, but it can help to understand the limitations of the acoustic modal analysis method as well. Let us take a three-dimensional spatial mesh in a Cartesian coordinate system, with unit mesh widths of $c\pi / L_x$, $c\pi / L_y$ and $c\pi / L_z$, respectively, parallel to the axes as shown in Fig.7c. Any vector from the origin to a nodal point of this lattice then corresponds to a particular eigen-frequency, because the length of this vector is equal to (22). Obviously, as the mode numbers are increasing, more and more combinations can result in almost identical vector lengths and thus closely spaced modal frequencies. If the number of modes per unit frequency is too high, the modes can no longer be distinguished in a proper way by experimental methods. Along with the matter of mode complexity as discussed above, these features of the acoustic eigenmodes can pose serious practical limitations to the feasibility of meaningful acoustic modal analyses.

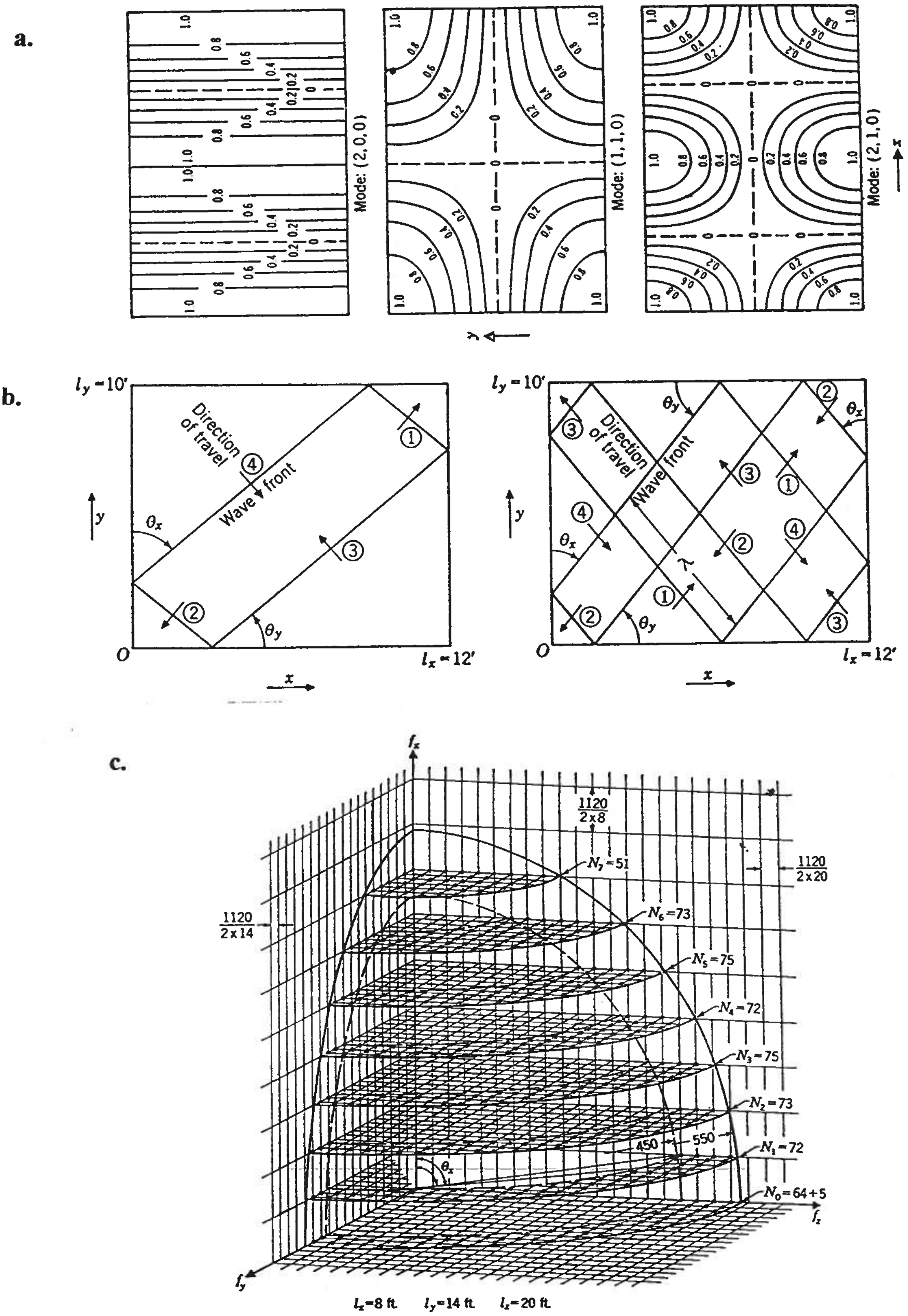

Fig.7. Possibilities of representation of acoustic modes in three-dimensional cavities. **a.** 2D demonstration of nodal lines, **b.** 2D demonstration of wavefronts, **c.** 3D demonstration of modal frequencies [15]

3.2. Modes in damped systems

The limited scope of this course notes does not allow the discussion of the eigenmodes of damped three-dimensional cases. The topics has been extensively investigated by Morse and Bolt in their basic work [6] and a good summary is given in the recent noise control textbook edited by L.L. Beranek and I.L. Vér [11,12]. We must be content here with making the remark that the qualitative results of a rigorous mathematical analysis are in complete agreement with those of the simple one-dimensional case: the damping results in complex modal frequencies with slightly changed real part and the mode shapes are always complex, for the reason detailed in paragraph 2.3. For more complex geometries the theoretical analysis is usually not feasible, and one has to resume numerical, Finite Element or Boundary Element methods.

3.3. Forced field in undamped, continuous acoustic systems

In reality, all modal analysis tests are based on measured forced responses, from which the eigenmodes are extracted, and acoustic modal analysis is no exception. It is therefore useful to derive the forced response of a general 3D acoustic cavity [8,13]. Note that this derivation will also be repeated for mechanical systems in paragraph 5.1.3. and as we will see, the results are in complete equivalence.

Let us consider a simple, elementary case: a point monopole of sinusoidal volume velocity, placed in point $\vec{r}_0$ of a closed space, surrounded by perfectly rigid (i.e., non-dissipative) boundaries. The governing equation in this case is [20]

$$\left(\nabla^2 + k^2\right) p = -\dot{q}\,\delta(\vec{r} - \vec{r}_0) \tag{23}$$

The solution to this inhomogeneous equation is the Green's function, depending both on the field point $\vec{r}$, the source point $\vec{r}_0$ and the frequency ω:

$$p(\vec{r}) = g(\vec{r}, \vec{r}_0, \omega)$$

Moreover, the Green's function has to satisfy the boundary condition $\partial g / \partial \vec{n} = 0$, too.

As a usual technique, the Green's function can be assumed to be a series expansion of the normal modes of the closed space such as

$$g(\vec{r}, \vec{r}_0, \omega) = \sum_m a_m \, \psi_m \tag{24}$$

where ψ_m is a solution to the homogeneous Helmholtz equation

$$\left(\nabla^2 + k_m{}^2\right) \psi_m = 0 \tag{25a}$$

Let ψ_n be another solution

$$\left(\nabla^2 + k_n{}^2\right)\psi_n = 0 \tag{25b}$$

Multiplying Eq.(25a) by ψ_n and Eq.(25b) by ψ_m and summing we obtain:

$$\psi_m\left(\nabla^2 + k_n{}^2\right)\psi_n - \psi_n\left(\nabla^2 + k_m{}^2\right)\psi_m = 0$$

But $\psi_m \nabla^2 \psi_n - \psi_n \nabla^2 \psi_m$ is the divergence of $\psi_m \nabla\psi_n - \psi_n \nabla\psi_m$. Thus, integration over the volume and application of Gauss' theorem yields :

$$\oiint_S (\psi_m \nabla\psi_n - \psi_n \nabla\psi_m) dS + (k_n^2 - k_m^2) \iiint_V \psi_n \psi_m \, dV = 0.$$

As $\Delta\psi_n$ and $\Delta\psi_m$ equal zero on the boundary surface S, one gets:

$$(k_n^2 - k_m^2) \iiint_V \psi_n \psi_m \, dV = 0$$

As $k_n \neq k_m$, this means that

$$\iiint_V \psi_n \, \psi_m \, dV = 0, \tag{26}$$

what can be interpreted as the orthogonality relation of acoustic eigenmodes.

The series of Eq.(24) is now substituted to Eq.(23). Making use of Eq.(25b) we have

$$\nabla^2 g + k^2 g = \sum_m a_m\left(k^2 - k_m^2\right)\psi_m = -\dot{q}\,\delta(\bar{r} - \bar{r}_0).$$

Multiplying both sides by ψ_n and integrating over the volume, one gets:

$$\sum_m a_m (k^2 - k_m^2) \iiint_V \psi_n \psi_m \, dV = \iiint_V -\dot{q}\, \psi_n\, \delta(\bar{r} - \bar{r}_0) \, dV.$$

Using the orthogonality of the modes and the definition formula of the Dirac function, we obtain

$$a_n (k^2 - k_n^2) \Lambda_n = -\dot{q}\; \psi_n(\bar{r}_0),$$

whereby

$$\Lambda_n = \iiint_V \psi_n^2 \, dV.$$

As a consequence, the weighting factors a_m equal

$$a_m = \frac{\dot{q}\psi_m(\vec{r}_0)}{\Lambda_m(k_m^2 - k^2)},$$

resulting in

$$g(\vec{r},\vec{r}_0,\omega) = \sum_m \dot{q}\,\frac{\psi_m(\vec{r})\,\psi_m(\vec{r}_0)}{\Lambda_m\left(k_m^2 - k^2\right)} \tag{27}$$

This equation is rather important for later developments: the forced response of an acoustic system is expressed by its normal modes. Note that the equation is symmetric in the source and response position, caused by and, at the same time, representing the reciprocity of acoustic systems.

4. Modes in vibro-acoustic systems

So far we have confined ourselves to purely acoustical systems, i.e., to cases where the mutual interaction between the acoustic fluid and the enclosing mechanical structure could be neglected. This assumption does not hold in a number of practical cases. Light mechanical structures submerged in or containing dense fluids, e.g. water, are typical examples, but a number of airborne acoustic problems can also be encountered in practice. In these cases we speak of *vibro-acoustic* or *coupled* modes. (Note that the expression "coupled modes" is not unambiguous in the literature. Those modes which are characterised by closely spaced modal frequencies and therefore it is difficult to distinguish them by means of experimental modal analysis methods, are also often referred to as "coupled modes" in structural dynamics.)

In fact, the analysis of vibro-acoustic systems requires the simultaneous solution of a mechanical and an acoustical eigenvalue problem, allowing for the interactions between them as well. This common analysis is not easy to perform by means of analytical methods in most of the cases, unless the geometry is very simple. The detailed analysis of such a system, namely, a double wall is discussed in a reference paper of this course notes [30], therefore this topic is not treated here any further.

5. Methods and tools of experimental modal analysis

Before we tackle the practical aspects of acoustic modal analysis, it is worth to survey the correspondence of mechanical and acoustic systems in analytical terms. This analogy is known for a long time [14-17], but nowadays is again of increased importance due to the availability and increasing use of structural analysis systems and methods for acoustic problems.

5.1. Analogies between acoustic, mechanical and electrical systems

5.1.1. Lumped parameter acoustic elements

Our analysis, presented in paragraph 2.2.1., is based on the assumption that the dimensions of the considered acoustic system is negligible with respect to the wavelength in all but one direction. This restriction can be fully extended in all of the directions, and this way one can come to the notion of the *lumped* or *concentrated parameter acoustic elements*.

Without going into much details of the elaboration (a good summary can be found e.g. in [15]), one can obtain the definition equations as follows.

The *acoustic mass* is the ratio of the pressure and the rate of change of the volume velocity, that is, the volume acceleration caused by the pressure:

$$m_a \equiv \frac{p}{\dot{q}} = \frac{p}{\alpha} = \frac{\rho l}{A} \tag{28}$$

It is associated with a mass of air accelerated by a net force which acts to displace the gas without appreciably compressing it. In structural terms, an acoustic mass has one degree of freedom, namely, its velocity displacement.

The *acoustic capacity* is described by the *acoustic compliance*: the ratio between the volume displacement and the pressure:

$$c_a \equiv \frac{\int q\,dt}{p} = \frac{\xi}{p} = \frac{V}{\rho c^2} \tag{29}$$

It is associated with a volume of fluid that is compressed by a net force without appreciable average displacement of the centre of gravity of the volume.

Eventually, the *acoustic resistance* is the ratio of the pressure and the volume velocity, caused by the pressure:

$$r_a \equiv \frac{p}{q} \tag{30}$$

Comparing these basic equations with their mechanical and electrical counterparts, a complete analogy can be established (cf. table 1). Making use of this equivalence, real-life acoustic systems can be modelled by electric or mechanical models and the acoustic problem can readily be solved by well established methods of these fields (see e.g. in [29]). The principle is demonstrated in Fig.8. on the example of a simple vehicle exhaust system.

5.1.2. *Matrix description of acoustical and mechanical systems*

Continuing the analogy, just like the Kirchoff equations in electricity theory and the equations of motion in mechanics, the acoustic equations can also be summarised in a matrix form:

$$[m_a]\{\ddot{\xi}\} + [r_a]\{\dot{\xi}\} + \left[\frac{1}{c_a}\right]\{\xi\} = \{p\} \tag{31}$$

where, for the sake of consistency, we have used the volume displacement ξ instead of the volume velocity q.

Another kind of acoustic analogy can also be established, in which the volume velocity corresponds to force and pressure corresponds to velocity. Beranek refers to this equivalence as a "mobility type" analogy as opposed to the earlier one which he calls an "impedance type". Other authors use the terms "direct" and "inverse" instead of impedance and mobility analogy. As the names show, the first one is more straightforward and it is traditionally used for electroacoustical applications while the latter one is more useful in vibroacoustic applications. Table 1. below shows the comparison of the two.

In order to obtain the second formulation, recall the inhomogeneous form of the wave equation:

$$\nabla^2 p - \frac{1}{c^2}\ddot{p} = -\rho\dot{q} \tag{32}$$

Using the methods of discrete system theory and introducing damping terms, several researchers [see e.g. 17, 18, 19] have shown that Eq.(32) can be converted into a matrix form like

$$[A]\{\ddot{p}\} + [B]\{\dot{p}\} + [C]\{p\} = \{\dot{q}\} \tag{33}$$

Comparing Eqs.(31) and (33) with the usual form of the mechanical equations of motion [22]

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{f\} \tag{34}$$

the formal analogy between Eq.(31) or (33) and Eq.(34) becomes obvious.

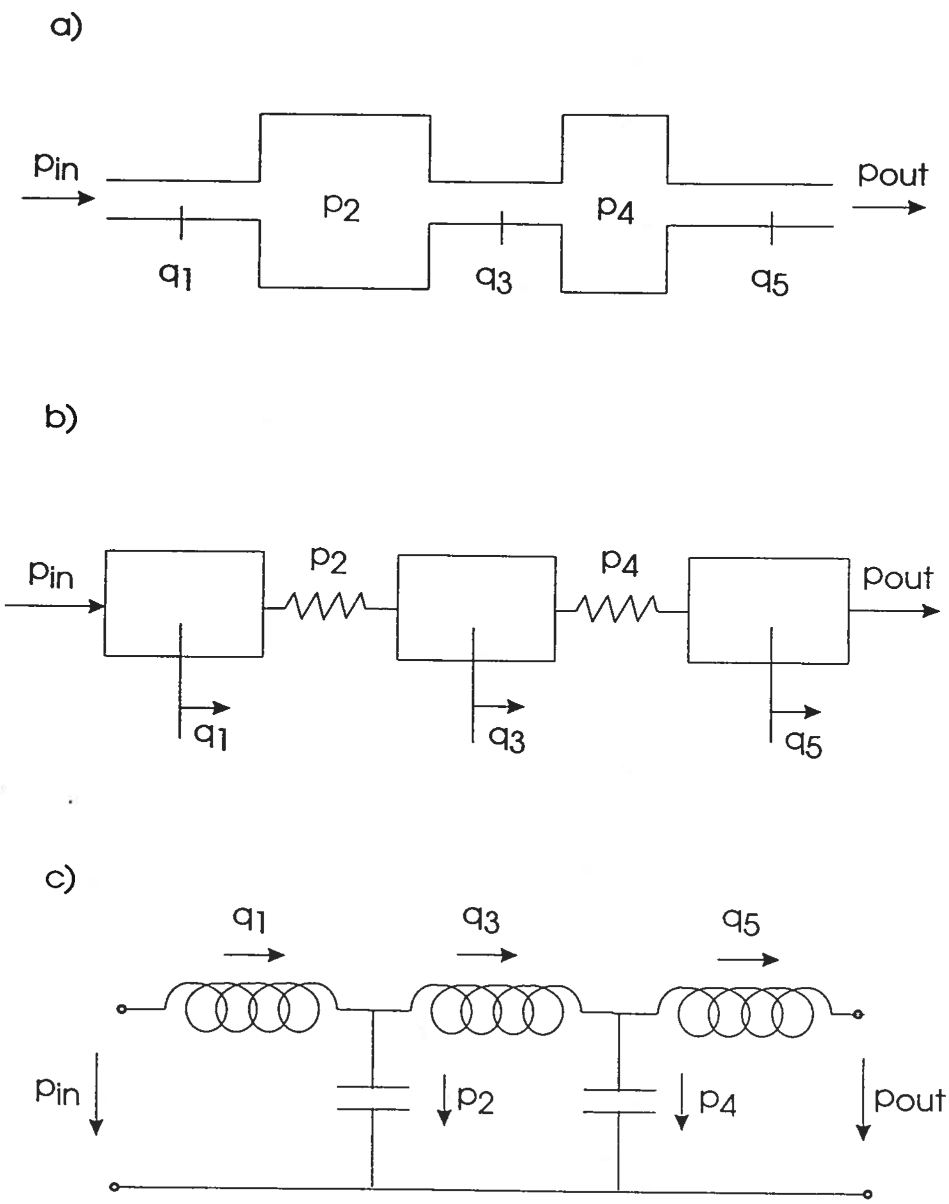

Fig.8. a. Acoustic model of an automotive exhaust sytem,
b. impedance-type mechanical analogous circuit,
c. impedance-type electrical analogous circuit

Table 1. Comparison of various analogies of lumped parameter elements [15]

Element	MECHANICAL ANALOGIES: Mobility type	MECHANICAL ANALOGIES: Impedance type	ACOUSTICAL ANALOGIES: Mobility type	ACOUSTICAL ANALOGIES: Impedance type
Infinite mechanical or acoustic impedance generator (zero mobility)	u	u	U	U
Zero mechanical or acoustic impedance generator (infinite mobility)	f	f	p	p
Dissipative element (resistance and responsiveness)	r_M, f, u	R_M, u, f	r_A, p, U	R_A, U, p
Mass element	M_M, f, u	M_M, u, f	M_A, p, U	M_A, U, p
Compliant element	C_M, f, u	C_M, u, f	C_A, p, U	C_A, U, p
Impedance element	z_M, f, u	Z_M, u, f	z_A, p, U	Z_A, U, p
Transformation element	Mech. to acous. (mobility type): f, $1:S$, p, u, U		Mech. to acous. (impedance type): u, $S:1$, U, f, p	

In principle there is nothing to prevent us from using either Eq.(31) or Eq.(33) as a basis to perform an acoustic modal analysis test, by using the methods originally developed to solve Eq.(34). However, there are some essential difference between them. The matrix description in Eq.(31) is based on the assumption that the system can be appropriately described by interconnecting individual lumped parameters, whereas Eq.(33) is free from this serious limitation. Moreover, the lumped parameter description as given in Eq.(31) needs the volume velocity to be measured as the response parameter and the system has to be excited by an ideal pressure source; neither is easy to realise in practice. Eq.(33) is therefore much more appropriate for experimental purposes. One also hase to note, nevertheless, that no direct physical meaning can be attributed to the matrix terms $[A]$, $[B]$ *and* $[C]$ while the use of Eq.(31) itself often gives a good physical insight into the nature of the problem. In summary, Eq.(31) is more appropriate for simple, quick-look calculations while the acoustic modal analysis experiments are based on Eq.(33).

5.1.3. *Forced response of discrete mechanical systems*

The derivation of Eq.(33), in other words, the discretisation of the continuous acoustic wave equation is based on the finite element formulation. However, this transition step can be avoided and the acoustic-mechanical analogy can be illustrated from a different aspect, if we proceed in opposite direction. Below we show that the forced response of a discrete mechanical system can be described in the same form as the forced response of a continuous acoustic system.

Assume an undamped multiple degree of freedom mechanical system, excited by sinusoidal forces. This system will be described by the matrix equation

$$\left[[K]-\omega^2[M]\right]\{x\}=\{f\} \tag{35}$$

The response vector $\{x\}$ can be calculated by inverting the matrix term on the left side, usually referred to as *system matrix* and denoted by $\lfloor B \rfloor$:

$$\{x\}=[B]^{-1}\{f\} \tag{36}$$

The inverse of the system matrix can be easier calculated, if we introduce the so called *principle* or *modal coordinates* by means of the transformation

$$\{x\}=[\Psi]\{q\} \tag{37}$$

where $[\Psi]$ is the *modal matrix:* a matrix whose columns are the modal vectors of the original system, and q stands throughout this paragraph for the new, modal coordinates [22]. The advantage of this coordinate transformation is that the original system of equation, consisting of general matrices, is decoupled and the new matrix equation contains diagonal matrices (denoted by the symbols $\langle\ \rangle$) only:

$$\left[\langle K\rangle-\omega^2\langle M\rangle\right]\{q\}=[\Psi]^T\{f\} \tag{38}$$

Introducing the notation $\langle B\rangle=\langle K\rangle-\omega^2\langle M\rangle$, the response of the system expressed in modal coordinates can be calculated, assuming that the frequency of excitation is different from any resonance frequency and thereby the diagonalized system matrix can be inverted:

$$\{q\}=\langle B\rangle^{-1}[\Psi]^T\{f\} \tag{39}$$

One can show [23] that the inverse of $\langle B\rangle$ will also be diagonal, containing the elements

$$\beta_i = \frac{1}{m_i\left(\omega_i^2 - \omega^2\right)} \tag{40}$$

Combining Eqs.(36), (37), (39) and (40) we eventually get

$$[B]^{-1} = [\Psi]\langle B\rangle^{-1}[\Psi]^T = \sum_{i=1}^{n} \frac{\left\{\Psi_r^{(i)}\right\}\left\{\Psi_e^{(i)}\right\}}{m_i\left(\omega_i^2 - \omega^2\right)} \tag{41}$$

One element, b_{re} of this matrix represents the response of the system in the rth DOF, excited by a unity force in the eth DOF

$$b_{re} = \sum_{i=1}^{n} \frac{\Psi_r^{(i)}\Psi_e^{(i)}}{m_i\left(\omega_i^2 - \omega^2\right)} \tag{41}$$

The expression Eq.(41) may be compared with that given in Eq.(27). The agreement justifies the use of the mechanical formulation for acoustic problems.

5.2. Experimental acoustic modal analysis

The first experimental modal analysis test, to the author's knowledge, has been reported in 1972, that time still without detailed theoretical background [24]. The first attempt to give a solid justification for the use of structural methods for acoustical problems stems from Nieter and Singh [25]. Their original technique, first applied for one-dimensional cases only, has later been extended to three-dimensional systems and an outline of the derivation of Eq.(33) was given [26]. A number of further publications about the topics are cited in the references [30-35].

As already mentioned shortly above, a wider spread of experimental acoustic modal analysis methods is impeded by a few principal and practical difficulties. One of these is that unlike in structural mechanics where almost all conceivable transfer functions can be readily measured in practice, in acoustics only the sound pressure can be determined reliably. Another difficulty, partly related to the previous one is the problem of how to provide appropriate excitation and reference signals therefrom. Below we overview the methods of experimental acoustic modal analysis and give some hints which may be able to overcome the difficulties encountered.

5.2.1. Basics and methods of the analysis

Consider a three-dimensional closed acoustic system with rigid or finite impedance but non-vibrating boundaries. (The modal analysis of vibro-acoustic systems goes beyond the scope of this paper.) The governing equation of the system can be written in the form

$$\nabla^2 p(\vec{r},t) - \frac{1}{c^2}\ddot{p}(\vec{r},t) = -\rho\dot{q}\,\delta(\vec{r}-\vec{r}_0) \qquad \text{=Eq.(32)}$$

Assume now that a number of point monopoles of known volume velocity are placed in the cavity and the sound pressure across the volume is sampled at an appropriate number of points. The continuous wave equation can then be substituted by its discrete equivalent

$$[A]\{\ddot{p}\} + [B]\{\dot{p}\} + [C]\{p\} = \{\dot{q}\} \qquad \text{=Eq.(33)}$$

Taking the Laplace-transform and assuming zero initial conditions we get

$$\left[s^2[A] + s[B] + [C]\right]\{p(s)\} = s\{q(s)\} \qquad (42)$$

As usual in structural dynamics, the inverse of the matrix term can be substituted by the frequency response matrix *H(s)*:

$$p(s) = [H(s)]\, s\{q(s)\} \qquad (43)$$

The frequency response matrix can in turn be expressed as a partial fraction expansion of modal parameters

$$[H(s)] = \sum_{i=1}^{n} \frac{Q_r\,\{\Psi\}_i\,\{\Psi\}_i^T}{s-\lambda_r} + \frac{Q_r^*\,\{\Psi\}_i^*\,\{\Psi\}_i^{*T}}{s-\lambda_r^*} \qquad (44)$$

Substituting now s by $j\omega$ and using Eq.(43) it becomes obvious, that the modal parameters of the system can be gained from the FRF measurements where the sound pressures across the volume are referenced to the volume velocities of the sources. In acoustic terms, the transfer impedances of the field have to be measured:

$$Z_{re}(\omega) = \frac{p(\omega)}{q(\omega)} = j\omega \sum_{i=1}^{n} \frac{(r_{re})_i}{j\omega-\lambda_i} + \frac{(r_{re})_i^*}{j\omega-\lambda_i^*} \qquad (45)$$

The expressions (42) to (45) are in complete analogy, up to the constant $j\omega$, with those being used in structural dynamics [22], therefore the usual structural methods and software packages can be used without modification.

5.2.2. *Equipment requirements and simplification possibilities*

In principle, no correct experimental acoustic modal analysis can be conducted without using a well-controlled volume velocity source. Unfortunately, such actuators are commercially not available. A few experimental systems have been reported on in the literature [25-28], out of which the converted acoustic driver method seems and actually has been found to be the most practicable [36].

Imagine an electrodynamic loudspeaker which is provided with a closed, sealed housing behind the diaphragm. The most obvious realisation could be to use a horn driver. Unfortunately, these loudspeakers are generally designed for high frequency sound reproduction and sometimes cannot radiate sufficient acoustic power in the frequency range relevant for acoustic modal analysis applications. A good quality medium-range loudspeaker with closed housing or, in case of even lower frequencies, a closed box loudspeaker unit may be helpful. (Note that the use of any bass-reflex boxes should be avoided, whatever attractive their low-frequency characteristics would be. The reflex opening of these units acts as another, unwanted and uncontrolled local radiator around resonance frequency and the resonance of the system can cause interpretation problems in the course of the analysis if not damped out sufficiently.)

If the back cavity's dimensions are considerably smaller than the wavelength, one can assume that the pressure is constant everywhere in the cavity. Then we have an acoustic capacity excited by the backward radiation of the diaphragm, causing a pressure in the cavity which can be calculated by means of Eq.(29):

$$p = q_{back}\frac{\rho c^2}{j\omega V} \qquad (46)$$

By measuring this pressure a good reference signal can be derived. In order to calibrate the whole system, another cavity of known volume can be connected to the loudspeaker and using the same formula the volume velocity, radiated forward can be related to the pressure measured in the back cavity.

This method has one single practical drawback, namely, that the pressure in the back cavity is very often too high, amenable to measurement. Another alternative is to measure the displacement of the diaphragm, implicitly assuming of course that the whole diaphragm moves with the same amplitude and phase. Substituting $q = Av$ in Eq.(46) it is easy to show that the pressure in the back cavity is proportional to the displacement.

To demonstrate the applicability of the technique, the cross section of an instrumented medium range speaker (Type Philips AD 50 060) is shown in Fig.9. along with the measured frequency response functions of the back pressure and the diaphragm displacement, measured by means of a Bentley proximity probe; both referenced to the input voltage of the loudspeaker. The similarity of the two FRF's supports that the diaphragm can indeed be considered as a rigid piston in the used frequency range.

If the analyst is interested in the modal frequencies and the mode shapes of the system only and a correct modal model is of no importance, the volume velocity source can be substituted by a simple loudspeaker. Then the reference signal can be taken directly from the input clamps of the speaker. (Needless to say, that the reference signal cannot be derived

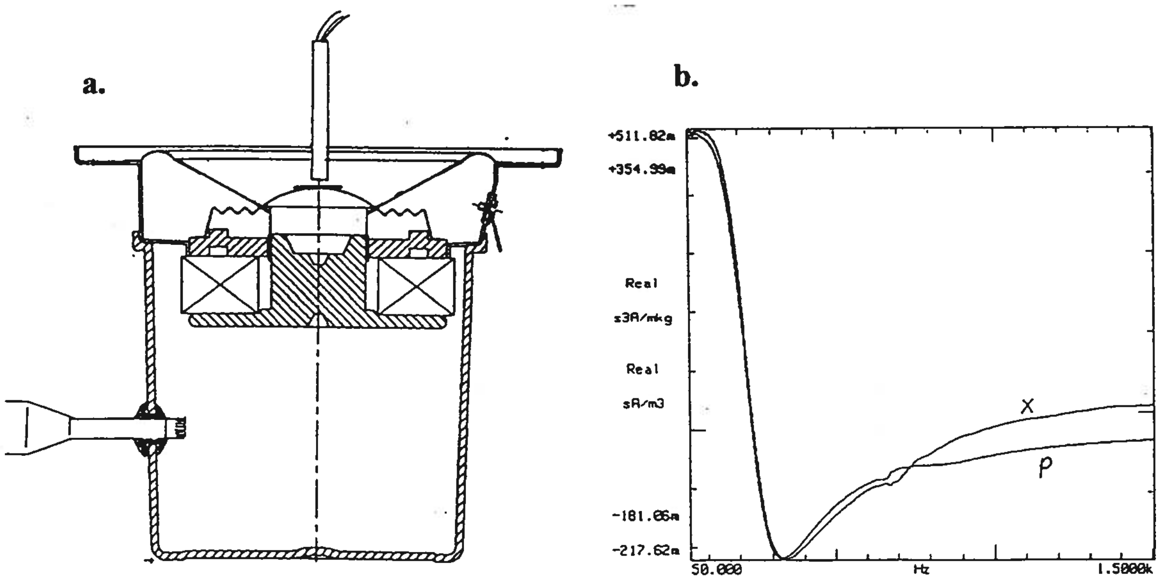

Fig.9. Controlled volume velocity source. **a.** Cross section of the loudspeaker, **b.** measured *p/u* and *x/U* frequency responses

by using a microphone in the close vicinity of the source. The sound pressure measured in any point of the volume is a response rather than an excitation signal. If one aims this way just to detect the modal frequencies only, it can happen that even strong normal modes will be missed if the microphone comes to a local maximum.) One has to be aware of the fact that in this case the loudspeaker itself becomes an element of the system to be investigated, and any possible resonances of the exciter appear in the analysis as supplementary modes. These false modes are not easy to distinguish from the actual modes of the system in case of strong damping. The same holds for the microphones. The solution is that before the actual test the frequency response function of the actuators and sensors have to be carefully checked.

As far as the sensors for the measurement of the responses are concerned, the difficulties are much smaller but a bit care is appropriate here, too. The acoustic field has to be sampled by using microphone positions which are closer than $\lambda/6$ (a general rule of thumb used for discrete acoustic methods). In case of large dimensions the number of microphone positions can run high. If one does not use an appropriate large number of parallel channels, the total measurement time can be too long, enabling the loudspeaker to heat up the air in the volume. Thereby the sound speed in air can change which in turn can cause the variation of the modal frequencies. In case of low damping, i.e., for sharp peaks in the FRF's, even a slight frequency shift can cause serious problems during postprocessing of the data. The problem can be overcome by using microphone arrays. Since, however, the costs of such arrays can be prohibitive if good quality measuring microphones are to be used, new types of low-cost microphones have been developed and commercialised, especially for acoustic modal analysis application [37]. We have found good results by using electret studio microphones, too.

By using the methods described above, a number of experiments have been carried out at the Structural Dynamics and Acoustics Laboratory of K.U.Leuven. The results of these experiments and case studies will be demonstrated in the oral presentation.

6. Summary

This course notes gave an overview over the physical reasons and conditions of formation of natural modes in acoustical systems. The notions related to acoustical modes and their inter-relations are discussed both in qualitative and in analytical terms for simple one-dimensional systems. Special attention was paid to the effects of non-rigid boundary conditions, being able to cause complex modal frequencies and complex mode shapes. The analysis was extended to the normal modes and forced field of three-dimensional systems.

The analogies between acoustical, mechanical and electrical systems are discussed in details. It is shown that the lumped parameter acoustic approach, based on the "direct" or "impedance type" analogy, has inevitable advantages for preliminary, quick-look calculations while the "inverse" or "mobility" type analogy is more appropriate for acoustic modal analysis experiments. The forced response of a continuous acoustic and a discrete mechanical system are calculated and found to be equivalent.

The outcome of theoretical considerations supports and justifies the engineering practice that those methods, originally developed for problems of structural mechanics, can also be applied in acoustics. However, high values and strong unproportionality of damping and the overlapping of modes at higher frequencies make the interpretation of results more difficult in acoustics than it is usual in structural dynamics. A further inconvenience is caused by the fact that the graphical interfaces of the presently available modal analysis systems are designed to animate spatial mechanical structures rather then scalar field variables

Eventually, some practical hints are given, how the equipment requirement of acoustic modal analysis can be met by simple methods.

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