Effects, interpretation and practical application of truncated SVD in the numerical solution of inverse radiation problems

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Abstract

The inversion of the numerical radiation problem can be an effective source identification tool, if vibration sensors are difficult to use but the sound field around the source can be measured in sufficient detail. The inversion is often burdened by numerical errors, which can be reduced by regularization techniques such as singular value decomposition, SVD. The paper aims at investigating the physical meaning and interpretation of SVD, its effect on the accuracy (including the inevitable information loss) of the results. The treatment starts from basic mathematics, and the principles and notions applied are extended through well-known mechanical systems to the acoustical radiation/source identification problem. It is shown that a close analogy exists between the mathematical formulation of "modal matrix", the natural "modes" of a vibrating system and the "field" and "surface modes" of an acoustical system. The "field" and "surface" modes are illustrated on the example of a simulated experiment with a hypothetical subwoofer source.

1. Introduction

Singular Value Decomposition (SVD, see e.g. [1] or [2]) is a numerical technique which is nowadays routinely used for a number of problems such as solution of inverse problems, source identification by means of principal component analysis, transfer path analysis and the like. Even though the technique is useful and effective, the real content and meaning of the procedure is somewhat vague, the physical interpretation of the notions and quantities involved is often far from obvious.

This paper addresses the application of SVD for an inverse acoustic radiation problem, which is called inverse Boundary Element Method or numerical holography [3 to 6]. The sound field, radiated from a vibrating body can be calculated numerically by using the boundary element (BE) method. The inversion of the BE calculation procedure is apt for calculating the vibration of the source, given the sound field in sufficient number of field points and provided that the matrix to be inverted is well conditioned. If this latter condition is not fulfilled, SVD can still be resorted to, aimed at reducing errors in the calculation [7,8,9]. Nevertheless, reduction of errors is accompanied by inevitable information loss as well. The aim of the paper is to investigate, how SVD actually works for the investigated problem, what the effects of the applied methods are and, in general, how the numerical calculation can be interpreted in terms of physical – mechanical and acoustical - quantities.

2. Relationship of modes and SVD

2.1 Mathematical definitions: spectral decomposition, modal matrix, SVD

In mathematics, the formulation of modal matrix can be introduced as follows. In usual cases (for the sake of simplicity, the exact conditions will not be further discussed here) the matrix $\underline{\underline{A}}$ can be expressed by

$$\underline{\underline{A}} = \sum_{k=1}^{N} l_k \underline{u}_k \underline{v}_k^H \tag{1}$$

where $\underline{u}_k$ and $\underline{v}_k$ build a biorthogonal vector system ($\underline{v}_k^H \underline{u}_l = d_{kl}$). The vectors $\underline{u}_k$ and $\underline{v}_k$ can be collected in the matrices $\underline{\underline{U}}$ and $\underline{\underline{V}}$ respectively, the parameters l_k in the diagonal matrix $\underline{\underline{L}}$, and the matrix form of equation (1) is then

$$\underline{\underline{A}} = \underline{\underline{U}}\, \underline{\underline{L}}\, \underline{\underline{V}}^H \tag{2}$$

Since the vectors in $\underline{\underline{U}}$ and $\underline{\underline{V}}$ form a biorthogonal vector system, $\underline{\underline{V}}^H \underline{\underline{U}} = \underline{\underline{E}}$. This means that $\underline{\underline{V}}^H = \underline{\underline{U}}^{-1}$. Equation (2) takes thus the form

$$\underline{\underline{A}} = \underline{\underline{U}}\, \underline{\underline{L}}\, \underline{\underline{U}}^{-1} \tag{3}$$

which is called spectral decomposition of the matrix $\underline{\underline{A}}$. $\underline{\underline{U}}$ has a special name, it is called modal matrix. If one multiplies equation (3) from right by $\underline{\underline{U}}$, a new form can be obtained:

$$\underline{\underline{A}}\, \underline{\underline{U}} = \underline{\underline{U}}\, \underline{\underline{L}} \tag{4}$$

or in a more visible form

$$\underline{\underline{A}}\left[\underline{u}_1 \cdots \underline{u}_k\right] = \left[\underline{u}_1 \cdots \underline{u}_k\right] \begin{bmatrix} \underline{l}_1 & & \\ & \ddots & \\ & & \underline{l}_k \end{bmatrix} \tag{5}$$

This is equivalent to the expression

$$\underline{\underline{A}}\,\underline{u}_i = l_i \underline{u}_i \tag{6}$$

for every $i=1...k$. From this form, we can clearly see that the resulting modal matrix $\underline{\underline{U}}$ of the spectral decomposition of matrix $\underline{\underline{A}}$ contains the eigenvectors of $\underline{\underline{A}}$.

With the definition of modal matrix, the singular value decomposition can be defined:

$$\underline{\underline{A}} = \underline{\underline{U}}\,\underline{\underline{\Sigma}}\,\underline{\underline{V}}^H , \tag{7}$$

where $\underline{\underline{S}}$ is a diagonal matrix containing the so called singular values, which are the eigenvalues of $\underline{\underline{A}}^H\underline{\underline{A}}$, $\underline{\underline{U}}$ is the modal matrix of $\underline{\underline{A}}\underline{\underline{A}}^H$, $\underline{\underline{V}}$ is the modal matrix of $\underline{\underline{A}}^H\underline{\underline{A}}$. It can be proven that the decomposition exists and is unique for all kind of matrices, and $\underline{\underline{U}}$ and $\underline{\underline{V}}$ are orthogonal. The elements of $\underline{\underline{S}}$ are ordered, the first (left upper) element is the highest, the last is the lowest.

The condition number k of $\underline{\underline{A}}$ is defined by the quotient of the largest and smallest singular value. Omitting some of the smaller singular values and the associated vectors, a better-conditioned matrix is achieved, which means a matrix of smaller condition number. This method – called truncated singular value decomposition, T-SVD – is often used in the solution of ill-conditioned matrix equations. The reason is proven in the matrix theory literature [see e.g. in 1]: it can be shown that to solve a matrix equation by using the direct way (applying the pseudo-inverse of $\underline{\underline{A}}$) is not advantageous if the condition number of $\underline{\underline{A}}$ is high, because the errors are amplified to an unacceptable extent. In order to keep errors between limits, some regularisation methods such as the truncated singular value decomposition are required.

2.2 Natural modes of mechanical systems

For the sake of compeleteness we recall that the natural modes of a mechanical system are closely related to the mathematical formulation described above. A governing system of equations of a simple undamped mechanical system can be written in the form

$$\left(\underline{\underline{K}} - w^2 \underline{\underline{M}}\right)\underline{x} = \underline{f} \tag{8}$$

where $\underline{f}$ is the input load (force), $\underline{x}$ is the displacement vector, $\underline{\underline{K}}$ is the stiffness and $\underline{\underline{M}}$ the mass matrix, and w is the angular frequency. The non-trivial solutions of equation (8) for $\underline{f}=0$ results in the modes of the mechanical system:

$$\left(\underline{\underline{M}}^{-1}\underline{\underline{K}}\right)\underline{x} = w^2\,\underline{x} \tag{9}$$

If one collects all the vectors $\underline{x}$ in a matrix $\underline{\underline{X}}$, and all the frequencies w^2 in $\underline{\underline{W}}$:

$$\left(\underline{\underline{M}}^{-1}\underline{\underline{K}}\right)\underline{\underline{X}} = \underline{\underline{X}}\,\underline{\underline{O}} \tag{10}$$

If the inverse of $\underline{\underline{X}}$ exists, then by multiplying from right we get

$$\left(\underline{\underline{M}}^{-1}\underline{\underline{K}}\right) = \underline{\underline{X}}\,\underline{\underline{O}}\,\underline{\underline{X}}^{-1} \tag{11}$$

Now we see the same form as in the mathematical section, see Eq.(3), implying that the modal matrix of the spectral decomposition of $\underline{\underline{M}}^{-1}\underline{\underline{K}}$ does contain the modes of the mechanical system indeed.

2.3 Definition of "modes" in radiation acoustical systems

A general approach of a wide class of numerical acoustical problems can be described as follows. The discretised model of the studied object has to be created, and some points in the field have to be selected, where we would like to know the sound pressure for example (see Figure 1). Based on the discrete geometry and the governing sound field equations, the relation of the descriptors of the source and field points can be expressed by

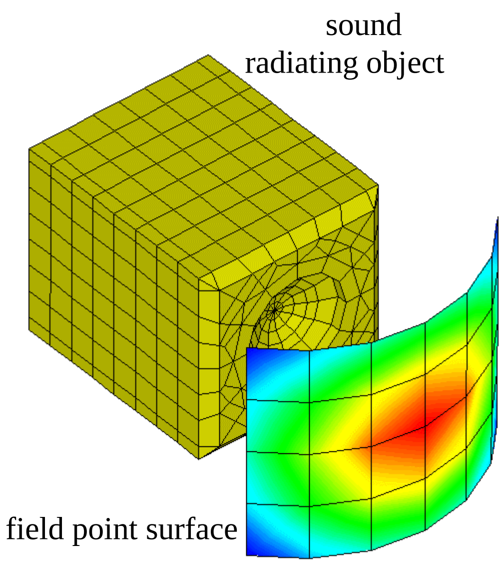


Figure 1. *Numerical model of a vibrating object and the corresponding sensing surface*

$$\underline{p} = \underline{\underline{c}}\,\underline{v}_s \tag{12}$$

a simple matrix equation, consisting of the normal surface velocity $\underline{v}_s$ as input, the sound pressure $\underline{p}$ as output and a frequency dependent transfer matrix $\underline{\underline{c}}$. The singular value decomposition of $\underline{\underline{c}}$ is given by

$$\underline{\underline{c}} = \underline{\underline{U}} < s > \underline{\underline{V}}^H \tag{13}$$

where $\underline{\underline{U}}^H \underline{\underline{U}} = \underline{\underline{V}}^H \underline{\underline{V}} = \underline{\underline{E}}$. Two new matrices can now be defined:

$$\underline{\underline{c}}\,\underline{\underline{c}}^H = \underline{\underline{U}} < |s|^2 > \underline{\underline{U}}^{-1} \tag{14}$$

and

$$\underline{\underline{c}}^H \underline{\underline{c}} = \underline{\underline{V}} < |s|^2 > \underline{\underline{V}}^{-1} \tag{15}$$

because

$$\underline{\underline{c}}\,\underline{\underline{c}}^H = \left(\underline{\underline{U}} < s > \underline{\underline{V}}^H\right)\left(\underline{\underline{U}} < s > \underline{\underline{V}}^H\right)^H = \underline{\underline{U}} < s > \underline{\underline{V}}^H \underline{\underline{V}} < s^H > \underline{\underline{U}}^H$$

and $\underline{\underline{U}}^H = \underline{\underline{U}}^{-1}$. The same derivation can be performed for $\underline{\underline{c}}^H\underline{\underline{c}}$. Thus it can be seen that the matrices $\underline{\underline{U}}$ and $\underline{\underline{V}}$ in the singular value decomposition of $\underline{\underline{c}}$ are modal matrices, in other words: they contain modes. The only question is now, what are they the modes of. In order to get a better insight, let us express the sound pressure by the singular value decomposition of the transfer matrix – in an additive form:

$$\underline{p} = \sum_{k=1}^{N} \underline{u}_k s_k \left(\underline{v}_k^H \underline{v}_s\right) = \sum_{k=1}^{N} \underline{u}_k a_k \tag{16}$$

s_k is here a scalar, and the product $\underline{v}_k^H \underline{v}_s$ is another one, so the sound pressure along the virtual surface defined in the field is the weighted sum of the modes $\underline{u}_k$. Therefore, we will introduce the name "field modes" for these quantities (see also [10] and [11]). Similarly, the normal velocity on the surface of the vibrating object can be expressed:

$$\underline{v}_s = \sum_{k=1}^{N} \underline{v}_k \frac{1}{s_k} \left(\underline{u}_k^H \underline{p}\right) = \sum_{k=1}^{N} \underline{v}_k b_k \tag{17}$$

In verbal form: the surface velocity is the weighted sum of the modes $\underline{v}_k$. As a result, in the following we will call them "surface modes". As it will be shown later on, the mathematical name "modal matrix", is not just a coincidence or expression of a formal analogy; its constituting vectors look very similar to the true mechanical natural modes of a vibrating body indeed.

3. Interpretation of SVD for a simulation experiment

The considerations described above lead to the conclusion that the singular value decomposition should be a kind of frequency decomposition. It couldn't be proved generally but a number of examples confirmed this assumption, including ours as well. In the acoustical case, the vectors $\underline{u}_k$ and $\underline{v}_k$ represent space-dependent quantities, hence the decomposition is a *spatial* frequency decomposition.

On the next pages an example of a hypothetical two-speaker subwoofer box of 0.42 × 0.25 × 0.21 m will be shown. Figure 2 shows the simple numerical model of the unit. It is assumed that the box is rigid everywhere, except for the two circular diaphragms, which move in-phase with velocity v_s = 1m/s. In front of the box two different field point surfaces can be seen, in order to have different condition numbers for the acoustical transfer matrix. The

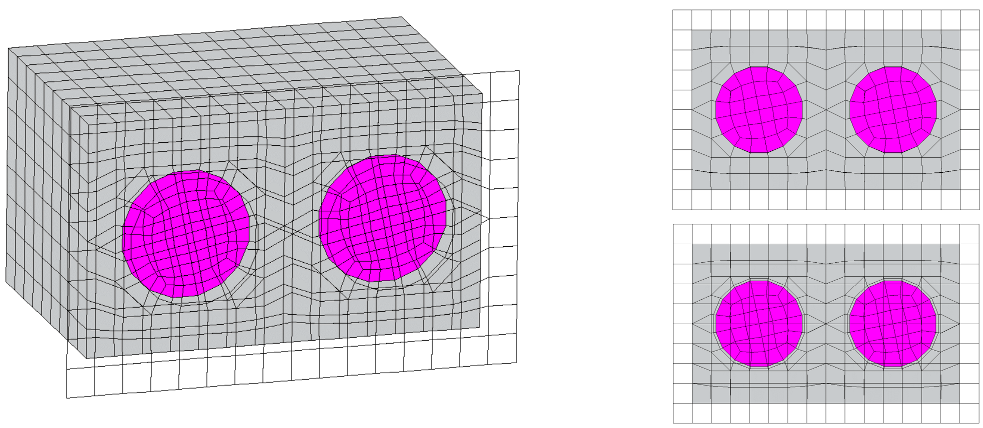


Figure 2. *Model mesh of the subwoofer box (right upper picture: optimal measurement mesh, right lower picture: non-optimal mesh)*

computations have been performed by using the Inverse Boundary Element method, of which the governing equation can be written as in equation (12). The computations presented here have been made for a typical operating frequency of 100 Hz.

3.1 Decomposition of the sound field (direct calculation) and the surface vibration (inverse calculation) from error-free data

As a first step of the analysis the optimal field point mesh will be used, which results in a better-conditioned acoustical transfer matrix. After singular value decomposition we obtain the two modal matrices $\underline{\underline{U}}$ and $\underline{\underline{V}}$. In Figure 3 the first 8 field modes (contained in $\underline{\underline{U}}$) can be seen. These are those elementary functions, the weighted sum of which composes the sound field (see equation (16)). Using a not complete summation of all modes, the sound pressure field can be calculated for different truncation values (Figure 4). As one can see, higher order modes hardly influence the sound field, so their omission – for a forward calculation problem – is not critical.

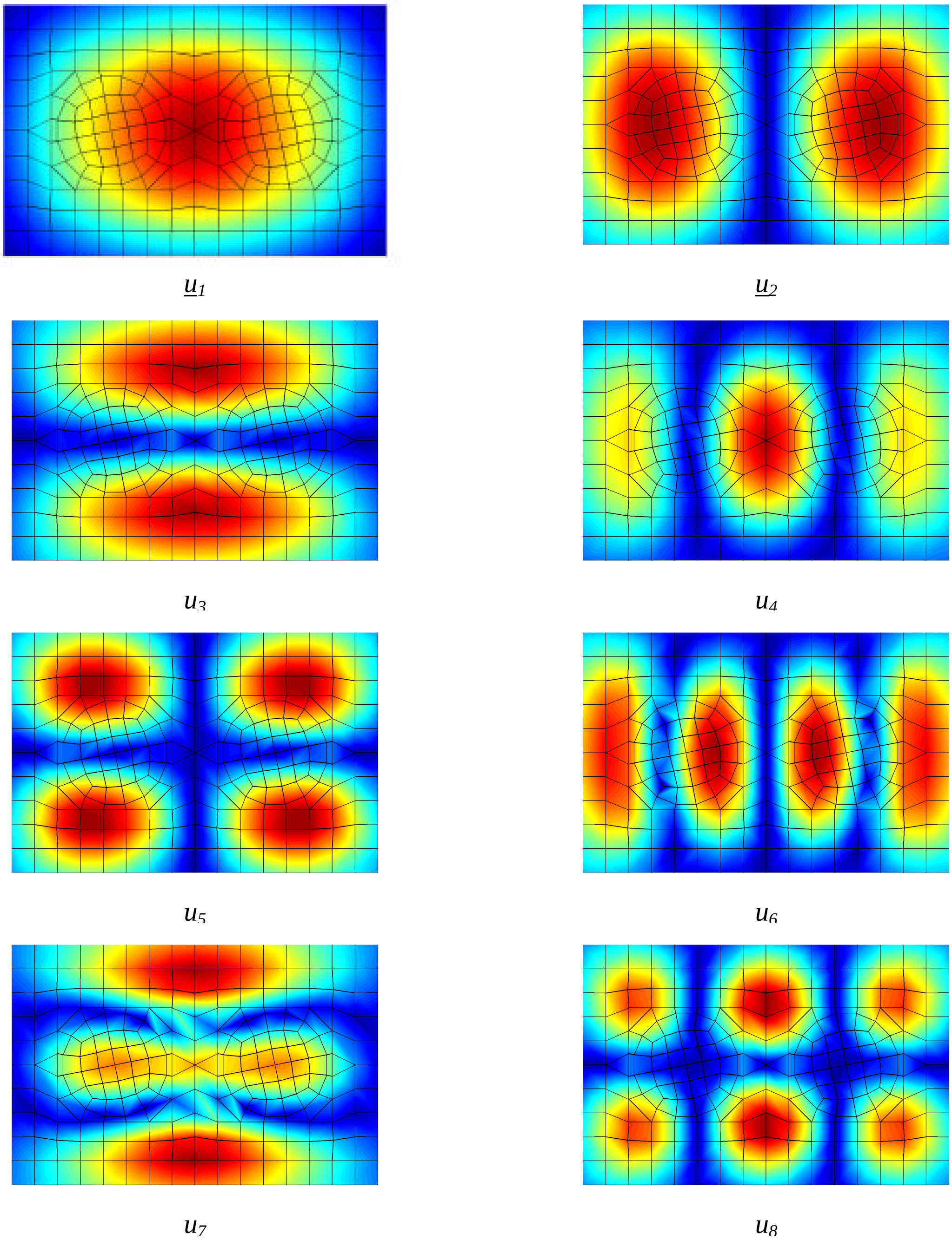
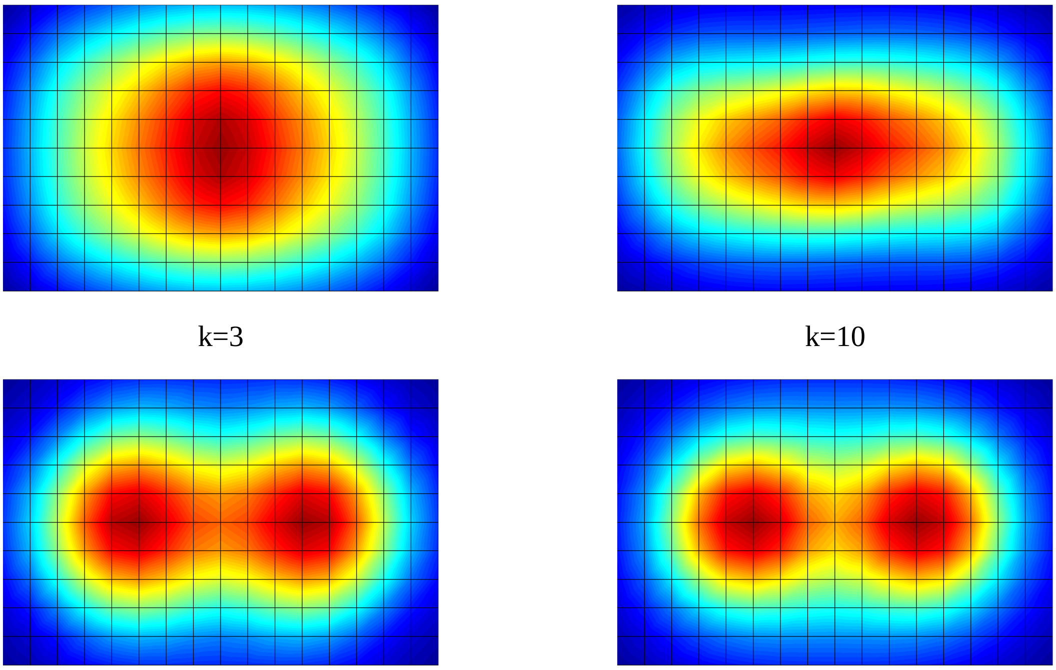


Figure 3. *The first eight field modes*

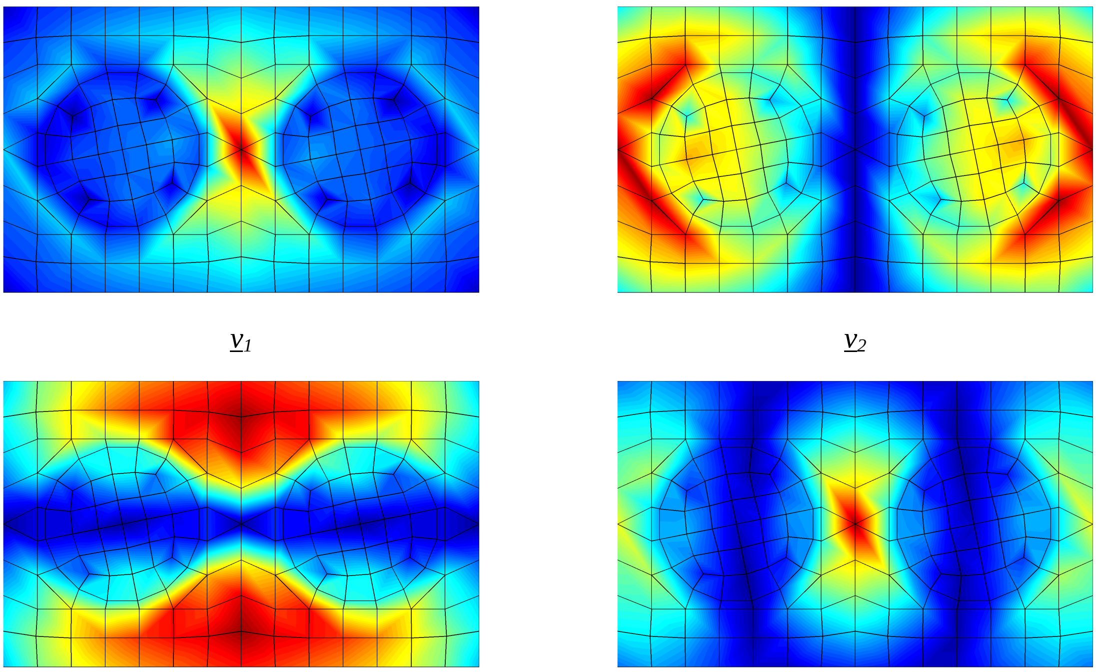


k=15 k=187

Figure 4. Pressure field along the measurement surface for different truncation values (k)

Not so for the inverse problem, where the rapid change of surface velocity – from the edge of the diaphragm to the rigid box – results in significant high spatial frequency components. Similar to the field modes, the surface modes also line up from lower to higher spatial frequencies in the order of increasing mode index (Figure 5).

This means that by limiting the summation of surface modes during the reconstruction of source velocities one will loose, or almost smooth out, rapid variations of surface velocity along the surface. This can clearly be observed in Figure 6 for different truncation parameters - still without any errors in the reconstruction.



$\underline{v}_3$ $\underline{v}_4$

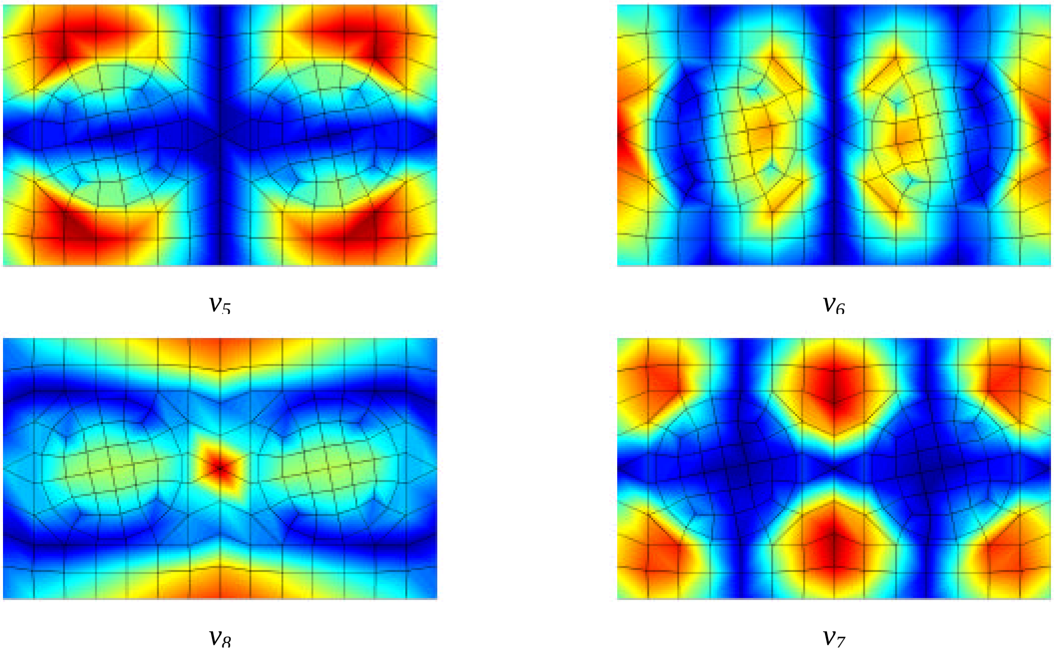


Figure 5. *The first eight surface modes*

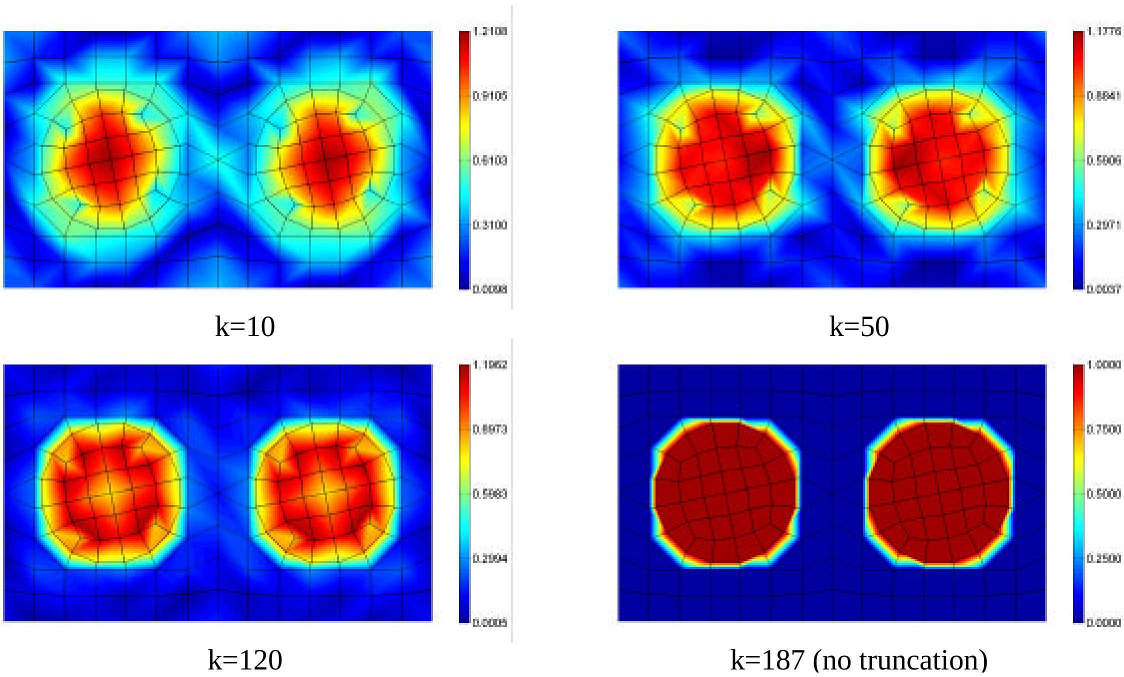


Figure 6. *Source velocity distribution for different truncation values (k)*

3.2 Inverse calculations from erroneous field data

As already mentioned earlier, regularisation methods are required to keep errors between limits. In order to see the potential of the method, in the following section the effects of noise will be discussed. Let us suppose uniformly distributed noise along the field point surface, which can e.g. originate from measurement or modelling errors. This means that the noise is a spatial white noise, so all the (spatial) frequencies are uniformly distorted. If the condition number of the acoustical transfer matrix lies in the order of 1000 (which is not too high for typical acoustical problems), the noise in the highest spatial frequency range (that is, for highest order modes) is amplified by some 60 dB with respect to the lowest spatial frequencies, thereby easily masking the useful information. Still with the optimal mesh, a comparison of reconstructed velocity distributions for two sorts of typical measurement noise can be seen in Figure 7.

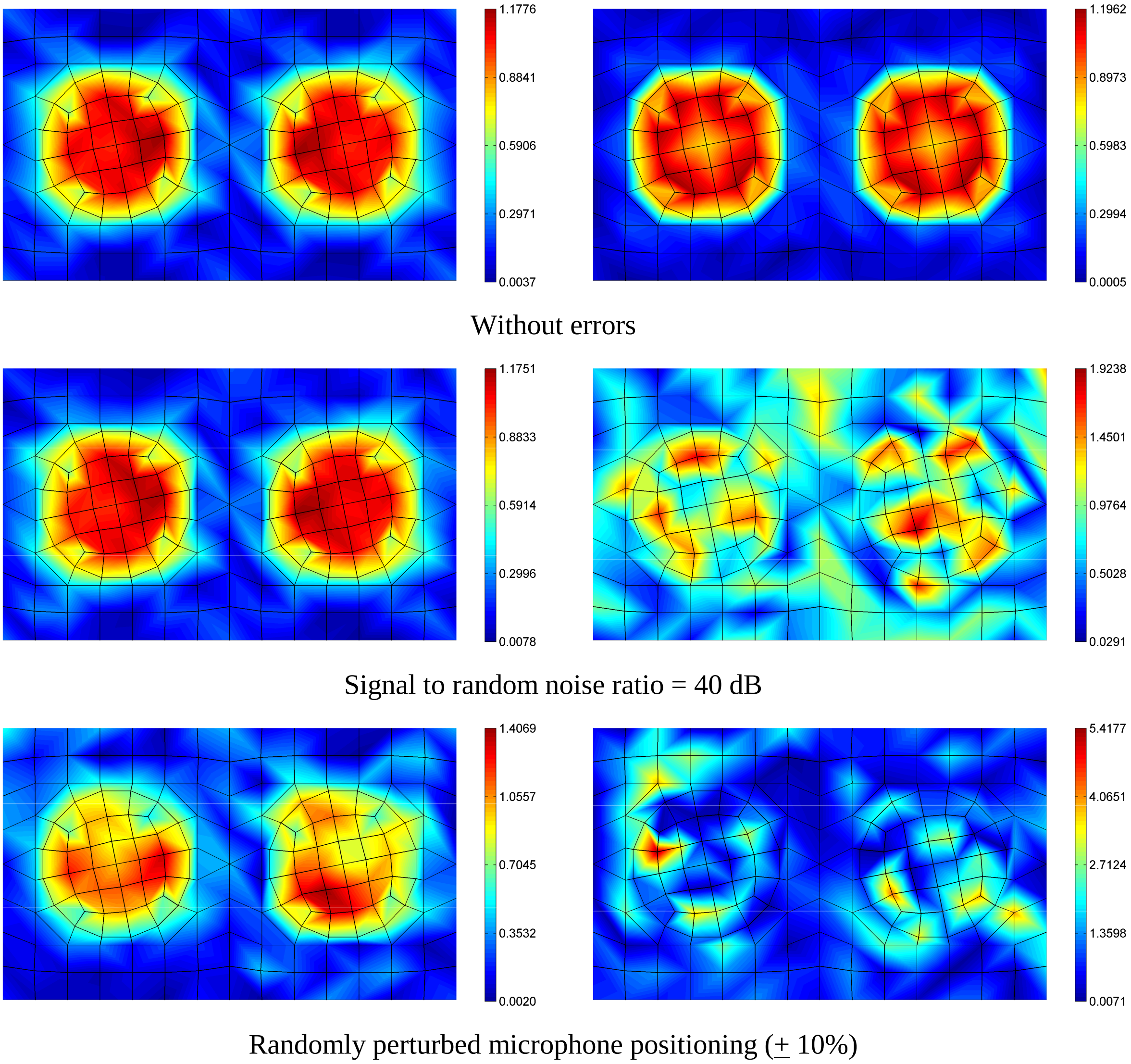


Figure 7. *Reconstructed velocity distributions for different kind of noises.*
Left *pictures: k=50 (nearly optimal truncation),* ***right*** *pictures: k= 120 (insufficient truncation)*

The last comparison will show, what happens if the condition number of the transfer matrix is different. In Figure 8 the singular values of the well conditioned and ill-conditioned matrix can be seen. It can clearly be observed that for a low truncation parameter (e.g. k=50) the two curves are nearly the same. If however the signal to noise ratio is high enough to take high frequency components also in account (e.g. k=150 or 160), the difference will be more and more significant. To visualise the phenomenon, we have performed a comparison for the two measurement meshes by assuming a signal to noise ratio of 60 dB (Figure 9).

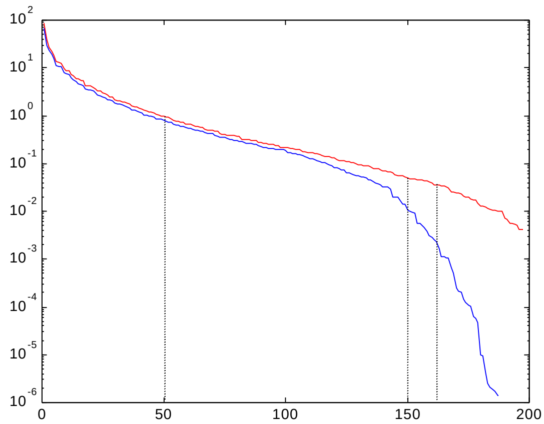


Figure 8. *Singular values for the optimal (upper curve) and non-optimal (lower curve) meshes*

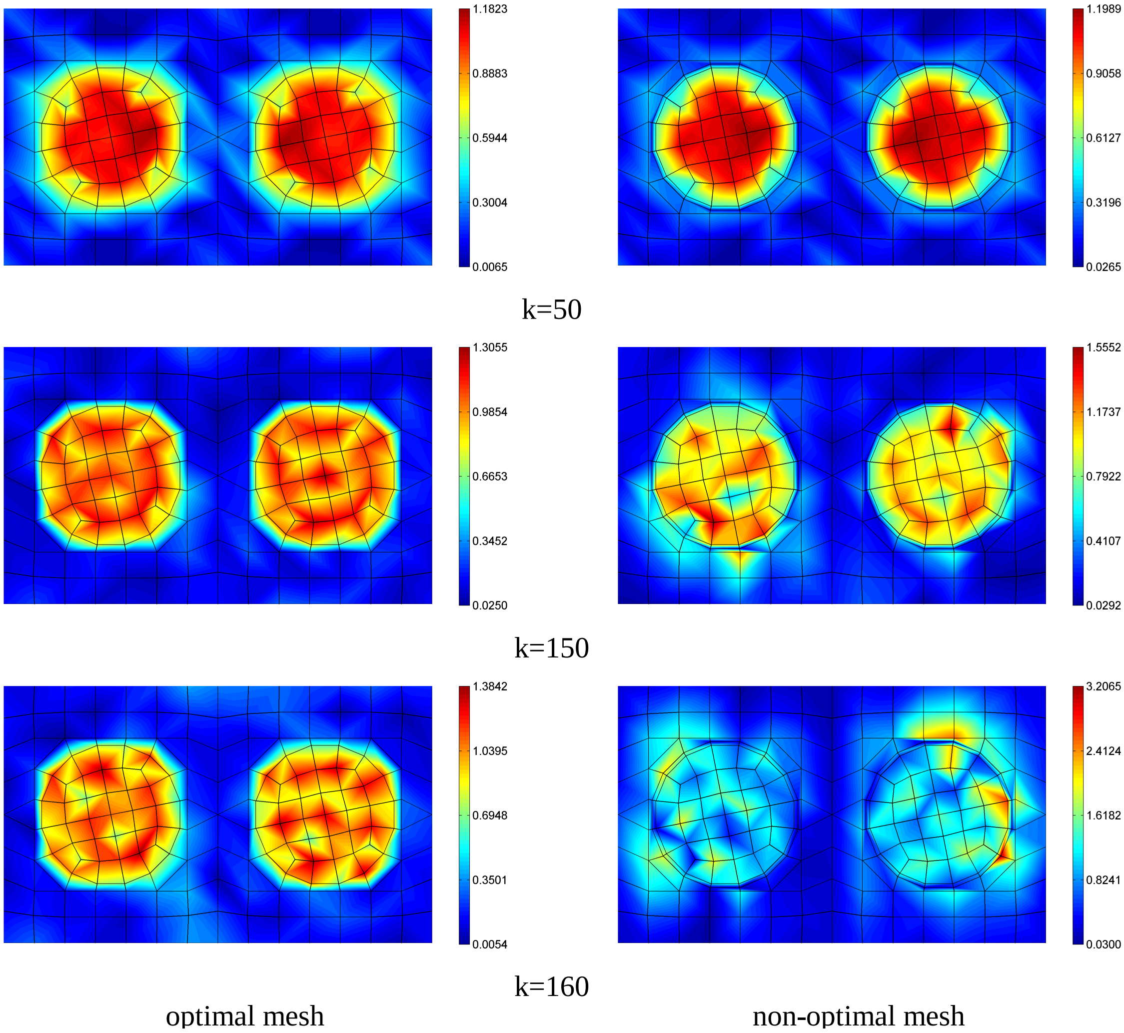


Figure 9. *Comparison of reconstructed velocity distributions for the optimal and non-optimal measurement meshes*

The effect of the SVD can then be interpreted as follows (see equation (16)): $\underline{\underline{V}}$ contains surface modes. These are excited by a particular surface velocity distribution $\underline{v}_s$. The surface modes excite the field modes ($\underline{u}_i$) in a weighted manner (s_i), which are similar to the modes of a homogeneous plate having a shape as defined by the field points. The largest contrast to real mechanical systems lies in the formulation of $\underline{\underline{V}}$. Due to the fact that surface modes are generated on the basis of the geometry only, material characteristics of various parts of an object are not taken in account.

4. Conclusions

The effects of the use of truncated singular value decomposition have been shown and a possible interpretation of the physical meaning of SVD has been given. Examples have shown that the surface velocity reconstruction is the best, when relatively low order surface modes are mostly excited. The cut-off parameter of the truncation can be set as a function of the signal to noise ratio. Note that the cut-off parameter determines the highest possible spatial frequency to be recovered and rapid transitions from low to high vibration amplitudes or vice versa are smoothed out accordingly.

5. Acknowledgement

Part of the works reported herein has been performed in the framework of the EUREKA Project 2122, "Methods and tools to address friction induced noise and vibration in brakes and wheels". The financial support of the Flemish Institution for Promotion of Scientific and Technological Research in Industry is gratefully acknowledged.

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