Fundamental solutions and Green's functions

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Theoretical Acoustics

Motivation

- The acoustical wave equation is a PDE (partial differential equation). Such equations are generally very hard to solve, analytical solutions only exsist in very simple, idealized cases.
- If the so-called *fundamental solution* of the PDE is known, instead of solving the PDE, we only need to calculate a convultion integral to get the solution of a particular problem.[1]
- In bounded domains, boundary conditions (BCs) must also be accounted for. The generalization of the fundamental solution that also satisfies some BC is called the *Green's function*.
- In simple cases, the Green's function can be constructed using the *method of images*.
- We will apply these construntions for calculating the static displacement of a membrane under spatially distributed load.

[1] Note that calculating an integral is a much easier task than solving a PDE.

Definitions

- Fundamental solution
  The solution $F(\mathbf{x}, \mathbf{x}_0)$ of the *linear* PDE

$$\mathcal{L}\left\{F(\mathbf{x}, \mathbf{x_0})\right\} = -\delta(\mathbf{x} - \mathbf{x}_0) \qquad \mathbf{x} \in \mathbb{R}^d$$

  Is called the *fundamental solution* of the PDE. Note that $\mathbf{x} \in \mathbb{R}^d$, which means that the domain is open.[2]

- Green's function
  The solution $G(\mathbf{x}, \mathbf{x}_0)$ of the *linear* PDE and a homogeneous BC defined over the whole boundary $\Gamma = \partial\Omega$

$$\begin{aligned} \mathcal{L}\left\{G(\mathbf{x}, \mathbf{x}_0)\right\} &= -\delta(\mathbf{x} - \mathbf{x}_0) \qquad && \mathbf{x} \in \Omega \subseteq \mathbb{R}^d \\ \mathcal{B}\left\{G(\mathbf{x}, \mathbf{x}_0)\right\} &= 0 && \mathbf{x} \in \Gamma \end{aligned}$$

  Is called the *Green's function* of the PDE and the respective BC. Note that domain can be bounded in this case.

[2] The minus sign on the right hand side is a matter of convention

Importance and usefulness

- Assume that we need to solve

$$\mathcal{L}\left\{u(\mathbf{x})\right\} = -g(\mathbf{x}) \qquad \mathbf{x} \in \mathbb{R}^d$$

- Statement: the solution is found as a convolution integral using the fundamental solution F

$$u = F * g \qquad u(\mathbf{x}) = \int F(\mathbf{x}, \mathbf{x}_0) g(\mathbf{x}_0)\,\mathrm{d}\mathbf{x}_0$$

- Proof:
  1. By definition: $\mathcal{L}\left\{F(\mathbf{x}, \mathbf{x}_0)\right\} = -\delta(\mathbf{x} - \mathbf{x}_0)$
  2. Multiply both sides by $g(\mathbf{x}_0)$ and integrate over the domain

$$\int \mathcal{L}\left\{F(\mathbf{x}, \mathbf{x}_0)\right\} g(\mathbf{x}_0)\,\mathrm{d}\mathbf{x}_0 = \int -\delta(\mathbf{x} - \mathbf{x}_0) g(\mathbf{x}_0)\,\mathrm{d}\mathbf{x}_0$$

  3. $\mathcal{L}$ acts on $\mathbf{x}$ and not $\mathbf{x}_0$, it can be moved outside the integral

$$\mathcal{L}\left\{\int F(\mathbf{x}, \mathbf{x}_0) g(\mathbf{x}_0)\,\mathrm{d}\mathbf{x}_0\right\} = -g(\mathbf{x})$$

Physical meaning

- The fundamental solution $F(\mathbf{x}, \mathbf{x}_0)$ is the response at the location $\mathbf{x}$ to a point source of unit strength located at $\mathbf{x}_0$.
- If we know $F(\mathbf{x}, \mathbf{x}_0)$ we can calculate the response to arbitrary source distributions $g(\mathbf{x})$ by using convolution.
- We are already familiar with linear electrical and mechanical state space models: in this case we are in the time domain, and the fundamental solution is the impulse response.
  If the system is time invariant, the response to a shifted input impulse, is also simply shifted in time.
- In a homogeneous medium, the operator $\mathcal{L}$ has constant coefficients. In these cases the fundamental solution is translation invariant, i.e., $F(\mathbf{x}, \mathbf{x}_0) = F(\mathbf{x} - \mathbf{x}_0)$.
  Thus, invariance in the time domain is analogous to a homogeneous medium in the space domain.

Finding the fundamental solution (an example)

- Find the fundamental solution of the Laplace equation in 2D

$$\nabla^2 F(\mathbf{x}, \mathbf{x}_0) = -\delta(\mathbf{x} - \mathbf{x_0}) \qquad \mathbf{x} \in \mathbb{R}^2$$

- Take first $\mathbf{x}_0 = \mathbf{0}$. As $u(\mathbf{x})$ is the field of a point source centered at the origin, we can expect that $F(\mathbf{x}, \mathbf{x}_0) = F(r)$. Thus, using the symmetric polar form of the laplacian:[3]

$$F''(r) + \frac{1}{r}F'(r) = \frac{-\delta(r)}{2\pi r} \qquad \left(\nabla^2 F = \frac{\mathrm{d}^2 F}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}F}{\mathrm{d}r}\right)$$

- We have

$$F''(r) + \frac{1}{r}F'(r) = 0 \qquad \forall r > 0 \quad \rightarrow \quad \frac{F''(r)}{F'(r)} = -\frac{1}{r}$$

- By integration we get

$$\ln F'(r) = -\ln r + c_0 \quad \rightarrow \quad F'(r) = \frac{c_1}{r} \quad \rightarrow \quad F = c_1 \ln r + c_2$$

[3] Note that the Dirac-delta in polar reads as: $\delta(x, y) = \delta(r)/(2\pi r)$.

- Any constant c_2 will satisfy the equation, so we take $c_2 = 0$.
- We can find c_1 by applying the divergence theorem[4]

$$\int_{\mathbb{R}^2} \nabla \cdot \nabla F(\mathbf{x}, \mathbf{x}_0) \, \mathrm{d}\mathbf{x} = \int_{\mathbb{R}^2} -\delta(\mathbf{x} - \mathbf{x}_0) \, \mathrm{d}\mathbf{x} = -1$$

- For all disks $B(R)$ with $R > 0$ we have

$$\int_{B(R)} \nabla \cdot \nabla F(\mathbf{x}, \mathbf{x}_0) \, \mathrm{d}\mathbf{x} = \int_{\partial B(R)} \mathbf{n}(\mathbf{x}) \cdot \nabla F(\mathbf{x}, \mathbf{x}_0) \, \mathrm{d}\mathbf{x} =$$

$$\int_{\partial B(R)} F'(R) \, \mathrm{d}\mathbf{x} = \int_0^{2\pi} F'(R) R \, \mathrm{d}\theta = 2\pi R F'(R) = -1$$

$$\rightarrow \quad F'(R) = \frac{-1}{2\pi R} \quad \rightarrow \quad F(r) = \frac{-\ln r}{2\pi} + c_2 \qquad (c_2 = 0)$$

- Thus, the fundamental solution of the Laplace equation in 2D:

$$F(r) = \frac{-\ln r}{2\pi} = \frac{\ln 1/r}{2\pi}$$

[4] It's instructive to carry out the integration in polar coordinates for $\delta(r)$

Green's function in bounded domains I.

- Once we have the fundamental solution, we can construct the Green's function for a bounded domain.

$$\begin{aligned} \nabla^2 G(\mathbf{x}, \mathbf{x}_0) &= -\delta(\mathbf{x} - \mathbf{x}_0) & \quad & \mathbf{x} \in \Omega \\ G(\mathbf{x}, \mathbf{x}_0) &= 0 & & \mathbf{x} \in \Gamma \end{aligned}$$

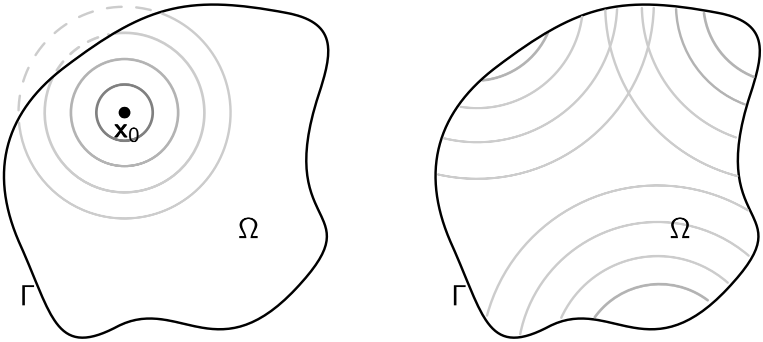
- We seek G as $G(\mathbf{x}, \mathbf{x}_0) = F(\mathbf{x}, \mathbf{x}_0) + v(\mathbf{x}, \mathbf{x}_0)$ with

$$\begin{aligned} -\nabla^2 v(\mathbf{x}, \mathbf{x}_0) &= 0 & \quad & \mathbf{x} \in \Omega \\ v(\mathbf{x}, \mathbf{x}_0) &= -F(\mathbf{x}, \mathbf{x}_0) & & \mathbf{x} \in \Gamma \end{aligned}$$

- Here $v(\mathbf{x}, \mathbf{x}_0)$ is a "correction term" that compensates the effect of F on the boundary.
- In general, this can be a very involving task. We arrive at a similar kind of BVP that we originally had.
- The superposition formula $G = F + v$ still gives us some important hint on the behavior of the Green's function.

Green's function in bounded domains II.

- Physical meaning of the superposition formula $G = F + v$



$F(\mathbf{x}, \mathbf{x}_0)$ $v(\mathbf{x}, \mathbf{x}_0)$

- $F(\mathbf{x}, \mathbf{x}_0)$ – response to a point source without boundaries
- $v(\mathbf{x}, \mathbf{x}_0)$ – effect of the boundaries, without source in Ω
- G – response to a point source with the effect of boundaries

Green's functions for the Laplace equation

- Let's construct the Green's function for some simple cases:
  1. 2D half-space with Dirichlet BC ($u(x, 0) = 0$)

     $$G(\mathbf{x}, \mathbf{x}_0) = F(\mathbf{x}, \mathbf{x}_0) - F(\mathbf{x}, \mathbf{x}_0^*) \qquad \mathbf{x}_0^* = (x_0, -y_0)$$

  2. 2D half-space with Neumann BC $\left(\frac{\partial u(x,0)}{\partial y}\Big|_{y=0} = 0 \right)$

     $$G(\mathbf{x}, \mathbf{x}_0) = F(\mathbf{x}, \mathbf{x}_0) + F(\mathbf{x}, \mathbf{x}_0^*) \qquad \mathbf{x}_0^* = (x_0, -y_0)$$

- Observe that adding the *mirror image sources* at the locations $\mathbf{x}_0^*$ satisfies the prescribed homogeneous BCs.
- This technique is often referred to as "method of images".

Example application – membrane I.

- Assume we want to calculate the shape of a membrane with unit radius under a steady force distribution.
  We have the BVP:

$$-S\nabla^2 u(\mathbf{x}) = g(\mathbf{x}) \qquad \mathbf{x} \in \Omega : \{|x| \leq 1\}$$
$$u(\mathbf{x}) = 0 \qquad \mathbf{x} \in \Gamma : \{|x| = 1\}$$

  (Where $S[\mathrm{N/m}]$ is the uniform tension of the membrane)
- First, let's look for the Green's function that satisfies

$$\nabla^2 G(\mathbf{x}, \mathbf{x}_0) = -\delta(\mathbf{x} - \mathbf{x}_0) \qquad \mathbf{x} \in \Omega : \{|x| \leq 1\}$$
$$G(\mathbf{x}, \mathbf{x}_0) = 0 \qquad \mathbf{x} \in \Gamma : \{|x| = 1\}$$

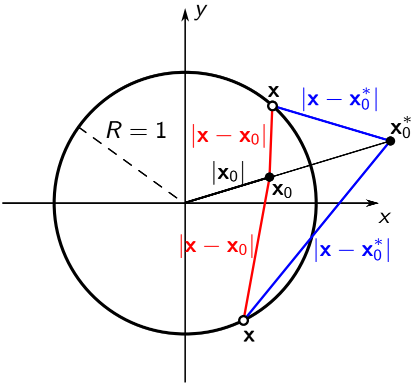
- Then, we can use the convolution form:

$$u(\mathbf{x}) = \frac{1}{S} g * G = \frac{1}{S} \int_\Omega g(\mathbf{x}_0) G(\mathbf{x}, \mathbf{x_0}) \, \mathrm{d}\mathbf{x}_0$$

Example application – membrane II.

- Construct the Green's function G by finding „image points"
- It turns out that for all $|\mathbf{x}_0| < 1$ and all $|\mathbf{x}| = 1$ there is an image point $\mathbf{x}_0^* = \mathbf{x}_0 / |\mathbf{x}_0|^2$ which satisfies

$$|\mathbf{x} - \mathbf{x}_0|^2 = |\mathbf{x}_0|^2 \, |\mathbf{x} - \mathbf{x}_0^*|^2$$



Example application – membrane III.

- By the above property of the image points we have

$$\ln|\mathbf{x}-\mathbf{x}_0| = \underbrace{\ln|\mathbf{x}_0| + \ln|\mathbf{x}-\mathbf{x}_0^*|}_{2\pi v(\mathbf{x},\mathbf{x}_0)}$$

- Thus, the term on the r.h.s. is $2\pi v(\mathbf{x},\mathbf{x}_0)$, with the Green's function $G(\mathbf{x},\mathbf{x}_0) = F(\mathbf{x},\mathbf{x}_0) + v(\mathbf{x},\mathbf{x}_0)$.
- The function $v(\mathbf{x},\mathbf{x}_0)$ is a compensating term for the fundamental solution $F(\mathbf{x},\mathbf{x}_0)$ such that the Green's function can satisfy the homogeneous Dirichlet BC on $|\mathbf{x}| = 0$.
- Finally, the Green's function for the membrane is found as

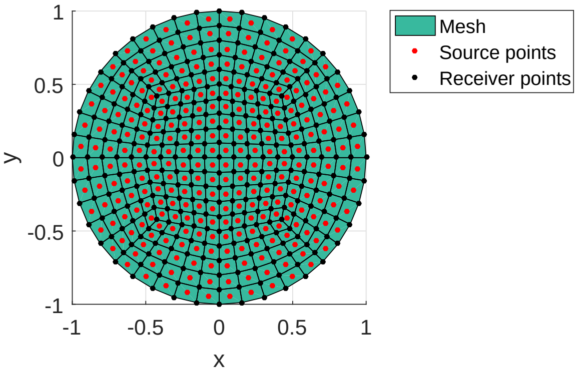
$$G(\mathbf{x},\mathbf{x}_0) = -\frac{1}{2\pi}\left(\ln|\mathbf{x}-\mathbf{x}_0| - \ln|\mathbf{x}-\mathbf{x}_0^*| - \ln|\mathbf{x}_0|\right)$$

Example application – membrane IV.

- Let's evaluate the displacement by using the convolution form

$$u(\mathbf{x}) = \frac{1}{S} g(\mathbf{x}_0) * G(\mathbf{x}, \mathbf{x}_0)\, \mathrm{d}\mathbf{x}_0$$

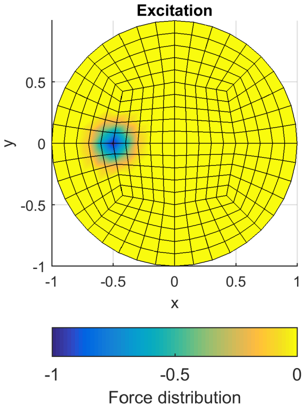
- We integrate numerically, and sum over "source points" ($\mathbf{x}_0$) to get the response in "receiver points" ($\mathbf{x}$)

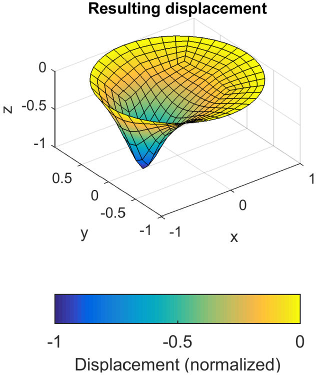


Example application – membrane V.

- Define the excitation as a truncated 2D Gaussian function

$$g(\mathbf{x_0}) = \begin{cases} -\mathrm{e}^{-\frac{|\mathbf{x}_e - \mathbf{x}_0|^2}{2\sigma_e^2}} & \text{if} \quad |\mathbf{x}_e - \mathbf{x}_0| < r_e \\ 0 & \text{otherwise} \end{cases}$$





Green's representation formula

- Green's identity
  Let u and v be smooth functions in Ω and $\mathbf{F} = u\nabla v - v\nabla u$.
  Thus, $\nabla \cdot \mathbf{F} = u\nabla^2 v - v\nabla^2 u$.
- By using the divergence theorem we get

$$\int_\Omega (u\nabla^2 v - v\nabla^2 u)\,\mathrm{d}\mathbf{x} = \int_{\partial\Omega} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) \mathrm{d}\mathbf{x}$$

  This is called *Green's identity*.
- Setting $u = G(\mathbf{x}, \mathbf{x}_1)$ and $v = G(\mathbf{x}, \mathbf{x}_2)$, with G being the Green's function of the Laplace equation,[5] we get the symmetry property for the Green's function

$$G(\mathbf{x}_1, \mathbf{x}_2) = G(\mathbf{x}_2, \mathbf{x}_1)$$

- In the following we will use the representation formula in the Boundary Element Method.

[5] satisfying either the Dirichlet or the Neumann BC