Boundary integral equation methods

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Motivation

- Using fundamental solutions and Green's functions solving the BVP is reduced to evaluating a convolution integral.
- However, constructing the Green's function of a problem with arbitrary geometry can be as hard as solving the original BVP.
- In the Boundary Integral Representation (BIR) the fundamental solution (free field Green's function) can be used.
- The physical interpretion of the BIR is also instructive: the *total field* is attained as the superposition of the *incident* and *scattered* fields.
- As an example application the membrane from the previous lecture is examined again.

The method of Boundary Integral Equations

General steps of the procedure

1. Find the fundamental solution[1] of the PDE
2. Derive the Boundary Integral Representation formula (BIR)
   2.1 Construct the weak form using an arbitrary test function
   2.2 Integrating by parts, shift operator to testing function
   2.3 Apply boundary conditions
   2.4 Apply Fundamental Solution as testing function
3. Solve boundary integral equation (BIE)
   3.1 Discretisation of boundary
   3.2 Discretisation of fields on the boundary
   3.3 Galerkin / Collocation
4. Express solution in internal points by applying the BIR

[1] Recall that the fundamental solution is the response to a point source in an infinite domain

Problem statement

Static displacement of an ideal membrane under distributed load

$$\text{PDE:} \qquad \nabla^2 u(\mathbf{x}) = -\frac{1}{S} g(\mathbf{x}), \quad \mathbf{x} \in \Omega \subset \mathbb{R}^2$$

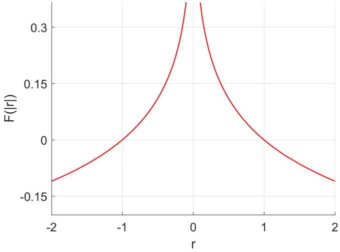
$$\text{BC:} \qquad u(\mathbf{x}) = 0, \quad \mathbf{x} \in \Gamma$$

- Fundamental solution

$$F(\mathbf{x} - \mathbf{x}_0) = -\frac{\ln |\mathbf{x} - \mathbf{x}_0|}{2\pi}$$

$$r = |\mathbf{x} - \mathbf{x}_0|$$

- Singularity at $r = 0$
- Physical meaning: the membrane cannot bear a concentrated force



Plot of the fundamental solution

Boundary integrals I.

- Let's construct the so-call *weak form* of the BVP

1. Test with testing function $\psi(x)$, and integrate $\int_\Omega \cdots \mathrm{d}\mathbf{x}$

$$\int_\Omega \psi(\mathbf{x})\nabla^2 u(\mathbf{x})\mathrm{d}\mathbf{x} = -\frac{1}{S}\int_\Omega \psi(\mathbf{x})g(\mathbf{x})\mathrm{d}\mathbf{x} \qquad \forall\psi(\mathbf{x})$$

- If we allow any test function $\psi(\mathbf{x})$, the solution of the integral equation must be the solution of the original PDE.

$$\int_\Omega \psi(\mathbf{x}) \underbrace{\left[\nabla^2 u(\mathbf{x}) + \frac{1}{S}g(\mathbf{x})\right]}_{\text{residual term}} \mathrm{d}\mathbf{x} = 0$$

- The residual term is non-zero if $u(\mathbf{x})$ does not perfectly satisfy the original PDE.
- We can choose a test function that emphasizes the residual in the neighborhood of an arbitrary point $\mathbf{x}_0$, such as $\delta(\mathbf{x} - \mathbf{x}_0)$.
- Thus, the residual must be zero in all points of the domain.
- Why is it called a "weak" form, then?

Boundary integrals II.

2. Integrate by parts until the operator (second derivatives) is shifted to the testing function
   - Use the relation: $\nabla \cdot (f\mathbf{g}) = \nabla f \cdot \mathbf{g} + f \nabla \cdot \mathbf{g}$
   - Apply integration: $\int_\Omega \nabla \cdot (f\mathbf{g})\, \mathrm{d}\mathbf{x} = \int_\Omega \nabla f \cdot \mathbf{g} \mathrm{d}\mathbf{x} + \int_\Omega f \nabla \cdot \mathbf{g} \mathrm{d}\mathbf{x}$
   - Use divergence theorem: $\int_\Omega \nabla \cdot (f\mathbf{g})\, \mathrm{d}\mathbf{x} = \int_\Gamma f\mathbf{g} \cdot \mathbf{n} \mathrm{d}\mathbf{x}$

   After integrating by parts twice, we get:

$$\int_\Omega \psi(\mathbf{x}) \nabla^2 u(\mathbf{x}) \mathrm{d}\mathbf{x} = \int_\Gamma \psi(\mathbf{x}) \frac{\partial u(\mathbf{x})}{\partial n} \mathrm{d}\mathbf{x} - \cancel{\int_\Gamma \frac{\partial \psi(\mathbf{x})}{\partial n} u(\mathbf{x}) \mathrm{d}\mathbf{x}} + \int_\Omega \nabla^2 \psi(\mathbf{x}) u(\mathbf{x}) \mathrm{d}\mathbf{x}$$

- At each integration by parts, one *boundary* integral is extracted from the volume integral. Finally, we get the original operator ∇^2 acting on the testing function.
- Weak form: weaker derivatives on the function $u(\mathbf{x})$

3. Apply BCs: in this case the 2nd term on the r.h.s. is zero

Boundary Integral Representation (BIR)

- Make the choice $\psi(\mathbf{x}) = F(\mathbf{x}, \mathbf{x}_0)$.
- As a result, the volume integral with the operator ∇^2 acting on F transforms into

$$\int_\Omega \nabla^2 F(\mathbf{x}, \mathbf{x}_0) u(\mathbf{x}) \mathrm{d}\mathbf{x} = -\int_\Omega \delta(\mathbf{x} - \mathbf{x}_0) u(\mathbf{x}) \mathrm{d}\mathbf{x} = \begin{cases} -u(\mathbf{x}_0) & \mathbf{x}_0 \in \Omega \\ -\frac{1}{2} u(\mathbf{x}_0) & \mathbf{x}_0 \in \Gamma \\ 0 & \text{otherw.} \end{cases}$$

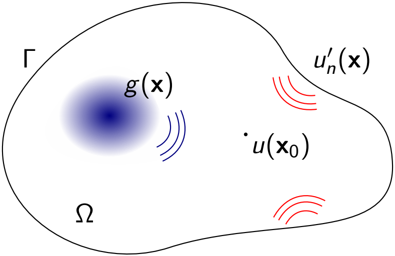
- Substituting and rearranging leads to the BIR:

$$\int_\Gamma F(\mathbf{x}, \mathbf{x}_0) \frac{\partial u(\mathbf{x})}{\partial n} \mathrm{d}\mathbf{x} + \frac{1}{S} \int_\Omega F(\mathbf{x}, \mathbf{x}_0) g(\mathbf{x}) \mathrm{d}\mathbf{x} = \begin{cases} u(\mathbf{x}_0) & \mathbf{x}_0 \in \Omega \\ \frac{1}{2} u(\mathbf{x}_0) & \mathbf{x}_0 \in \Gamma \\ 0 & \text{otherw.} \end{cases}$$

Physical interpretation of the BIR

Exploiting the symmetry:[2] $F(\mathbf{x}, \mathbf{x}_0) = F(\mathbf{x}_0, \mathbf{x})$

$$\underbrace{\int_\Gamma F(\mathbf{x}_0, \mathbf{x}) \frac{\partial u(\mathbf{x})}{\partial n} \mathrm{d}\mathbf{x}}_{\substack{\text{surface convolution} \\ \text{scattered field}}} + \underbrace{\frac{1}{S} \int_\Omega F(\mathbf{x}_0, \mathbf{x}) g(\mathbf{x}) \mathrm{d}\mathbf{x}}_{\substack{\text{volume convolution} \\ \text{incident field}}} = \underbrace{u(\mathbf{x}_0)}_{\substack{\text{total} \\ \text{field}}} \quad \mathbf{x}_0 \in \Omega$$



- Volume convolution $\rightarrow u_{\text{inc}}(\mathbf{x}_0)$
  Source: $g(\mathbf{x})$ in $\mathbf{x} \in \Omega$

- Surface convolution $\rightarrow u_{\text{scat}}(\mathbf{x}_0)$
  Source: $u'_n(\mathbf{x})$ on $\mathbf{x} \in \Gamma$

The total field is a superposition: $u(\mathbf{x}_0) = u_{\text{inc}}(\mathbf{x}_0) + u_{\text{scat}}(\mathbf{x}_0)$

[2] see also: Green's representation formula

Boundary Integral Equation (BIE)

- Let $\mathbf{x}_0$ approach the boundary Γ

$$u_{\text{inc}}(\mathbf{x}_0) + u_{\text{scat}}(\mathbf{x}_0) = 0$$

  (Note that the r.h.s. is zero in this special case because of the BC $u(\mathbf{x}) = 0$, if $\mathbf{x} \in \Gamma$)

$$\int_\Gamma F(\mathbf{x}_0, \mathbf{x}) \frac{\partial u(\mathbf{x})}{\partial n} \mathrm{d}\mathbf{x} = -u_{\text{inc}}(\mathbf{x}_0)$$

- Problem: $\frac{\partial u}{\partial n}$ is unknown on the boundary.
  Thus, this equation must be solved on the boundary.
- We look for an *approximate* solution using a numerical technique (a process "easily" executed by a computer)
- This technique is the Boundary Element Method (BEM)

Boundary Discretization

- Geometrical discretization: Discretize the boundary into disjunct boundary elements $\Gamma \approx \bigcup \Gamma_e$ ($\Gamma_i \cap \Gamma_j = \emptyset$, if $i \neq j$)
- The integral over Γ is the sum of integrals over elements Γ_e

$$\sum_{e=1}^{E} \int_{\Gamma_e} F(\mathbf{x}_0, \mathbf{x}) \frac{\partial u(\mathbf{x})}{\partial n} \mathrm{d}\mathbf{x} = -u_{\text{inc}}(\mathbf{x}_0)$$

- Function (or data) discretization: take the normal derivative of the displacement $\frac{\partial u}{\partial n} = \phi$ as constant over each element Γ_e

$$\sum_{e=1}^{E} \int_{\Gamma_e} F(\mathbf{x}_0, \mathbf{x}) \mathrm{d}\mathbf{x}\, \phi_e = -u_{\text{inc}}(\mathbf{x}_0)$$

  This may seem crude, yet often applied in practice.
- We now have E unknowns (ϕ_e, $e = 1, 2, \ldots E$) for the E elements. We also need E equations to get the solution.

BEM System of Equations

- Collocation: E independent equations by placing the receiver point $\mathbf{x}_0$ into the center of each element $\mathbf{x}_i$, $i = 1, \dots E$

$$\sum_{e=1}^{E} \int_{\Gamma_e} F(\mathbf{x}_i, \mathbf{x}) \mathrm{d}\mathbf{x}\, \phi_e = -u_{\text{inc}}(\mathbf{x}_i), \quad i = 1 \dots E$$

- The same in matrix form:

$$\mathbf{F}\phi = \mathbf{g} \qquad (\mathbf{F} : E \times E; \quad \phi \text{ and } \mathbf{g} : E \times 1)$$

- Matrix and vector elements are attained as:

$$F_{ij} = \int_{\Gamma_j} F(\mathbf{x}_i, \mathbf{x}) \mathrm{d}\mathbf{x}$$

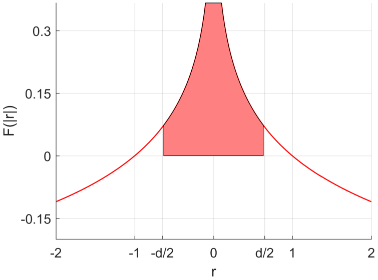
$$g_i = -u_{\text{inc}}(\mathbf{x}_i) = -\frac{1}{S} \int_{\Omega} F(\mathbf{x}_i, \mathbf{x}) g(\mathbf{x}) \mathrm{d}x$$

- The system Matrix $\mathbf{F}$
  - Full, not symmetric, (real-valued in this case)
  - Elements computed by (numerically) integrating the Fundamental Solution

Singular Integrals

- As the fundamental solution is singular, the diagonal matrix elements need to be handled separately
- Simple cases
  analytical integration

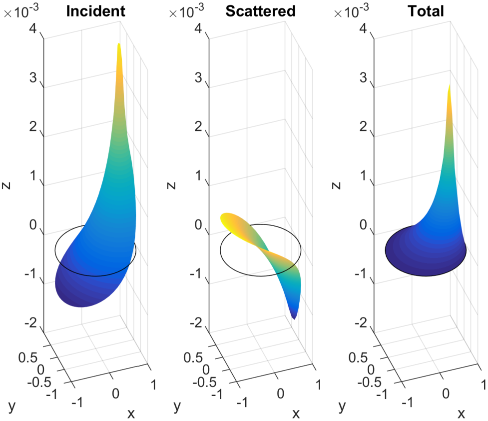
$$F_{ii} = \int_{\Gamma_i} F(\mathbf{x}_i, \mathbf{x}) \mathrm{d}\mathbf{x}$$
$$= \int_{-d/2}^{d/2} \frac{\ln 1/|x|}{2\pi} \mathrm{d}x$$
$$= \frac{d}{2\pi}(1 - \ln d/2)$$



(d: element size)
- Off-diagonal elements can be computed numerically

Example application

- Displacement of a membrane (concentrated force)



- Observe that the scattered field exactly compensates the incident field on the boundary
- Note that the solution is infinite in the point of excitation