The Boundary Element Method in Acoustics

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Theoretical Acoustics

Motivation

- Construction of the BEM for the Helmholtz equation
- Components of the Boundary Element Method (BEM)
  1. We have a BVP (= PDE + BC)
  2. We construct the boundary integral representation (BIR) using the fundamental solution of the PDE
  3. BIR is applied for the boundary points to get a boundary integral equation (BIE)
  4. BIE is discretized to get a linear system of algebraic equations. The discretization of the BIE is called the BEM.
  5. The discretized system is solved to get the unknown quantities on the boundary.
  6. Finally, the BIR can be utlizied to compute the radiated quantities (i.e., the sound pressure) in any point of the domain.

Frequency domain acoustics I.

- The wave equation for the acoustic pressure $p(\mathbf{x}, t)$ is

$$\nabla^2 p(\mathbf{x}, t) - \frac{1}{c^2}\frac{\partial^2 p(\mathbf{x}, t)}{\partial t^2} = -Q(\mathbf{x}, t) \qquad \mathbf{x} \in \Omega$$

  with $Q(\mathbf{x}, t)$ denoting the spatially distributed source term
- We take time-harmonic sources with angular frequency ω, due to linearity field quantities oscillate with the same frequency

$$\begin{aligned} p(\mathbf{x}, t) &= A(\mathbf{x}) \cos(\omega t + \phi(\mathbf{x})) \\ &= \mathrm{Re}\,\{A(\mathbf{x}) \exp(\mathrm{j}\omega t + \mathrm{j}\phi(\mathbf{x}))\} \\ &= \mathrm{Re}\,\{\underbrace{A(\mathbf{x})\mathrm{e}^{\mathrm{j}\phi(\mathbf{x})}}_{\hat{p}(\mathbf{x})}\mathrm{e}^{\mathrm{j}\omega t}\} \end{aligned}$$

- As ω is fixed, the *complex amplitude* $\hat{p}(\mathbf{x})$ describes the variation of p in space and time too. The complex amplitude conveniently contains both the amplitude and the phase.

Frequency domain acoustics II.

- The time derivation $\partial/\partial t$ is simply a multiplication by $\mathrm{j}\omega$ in the frequency domain. If $p \to \hat{p}$, then $\partial p/\partial t \to \mathrm{j}\omega\hat{p}$.
- This leads to the *Helmholtz equation*

$$\nabla^2 \hat{p}(\mathbf{x}) + k^2 \hat{p}(\mathbf{x}) = -\hat{Q}(\mathbf{x}) \qquad \mathbf{x} \in \Omega$$

  with $k = \omega/c$ denoting the wave number
- In case of BCs the linearized Euler equation is often used

$$\begin{aligned} \nabla p(\mathbf{x}, t) + \rho_0 \frac{\partial \mathbf{v}(\mathbf{x}, t)}{\partial t} &= \mathbf{0} \qquad && \mathbf{x} \in \Omega \\ \nabla \hat{p}(\mathbf{x}) + \mathrm{j}\omega\rho_0 \hat{\mathbf{v}} &= \mathbf{0} \qquad && \mathbf{x} \in \Omega \end{aligned}$$

- After scalar multiplication by the surface normal vector $\mathbf{n}(\mathbf{x})$

$$\frac{\partial \hat{p}(\mathbf{x})}{\partial n} + \mathrm{j}\omega \hat{v}_n = 0 \qquad \mathbf{x} \in \Gamma$$

- “Hat” notation is often omitted and simply p and $\mathbf{v}$ are used

Green's function for the Helmholtz equation

- For the Helmholtz equation the fundamental solutions can be constructed using Fourier transform, for example.
  This is interesting, but not discussed here in detail.
- We have the free field Green's functions (fundamental solutions) for the Helmholtz equation as

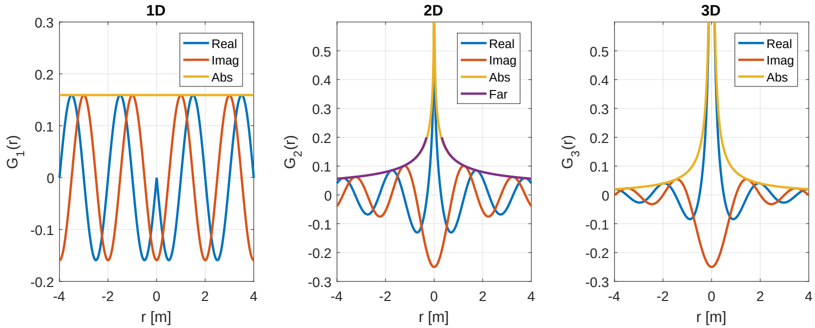
$$\text{1D:} \qquad G_1(x, x_0) = \frac{1}{2k\mathrm{j}} \mathrm{e}^{-\mathrm{j}kr}$$

$$\text{2D:} \qquad G_2(\mathbf{x}, \mathbf{x}_0) = -\frac{\mathrm{j}}{4} H_0^{(2)}(kr)$$

$$\text{3D:} \qquad G_3(\mathbf{x}, \mathbf{x}_0) = \frac{\mathrm{e}^{-\mathrm{j}kr}}{4\pi r}$$

  with $r = |\mathbf{x} - \mathbf{x}_0|$ in all cases.
  $H_0^{(2)}$ is the Hankel function of the second kind, zeroth order
- Note: in the limit $k \to 0$ we get the free field Green's functions of the Laplace equation

Plots of the Green's functions



- Common properties
  - Oscillation with period $\lambda = 2\pi/k$ (λ is the wavelength)
  - Derivative of real part discontinuous at $r = 0$
  - Imaginary part smooth in the whole domain
  - Decay $\propto r^{-(d-1)/2}$ (d is the number of dimensions)
  - In 2D and 3D the functions are singular at $r = 0$

BIE for the Helmholtz equation

- A.k.a. Kirchhoff–Helmholtz integral equation (KHIE)
- We have the inhomogeneous[1] Helmholtz equation as

$$\underbrace{\nabla^2 p(\mathbf{x}) + k^2 p(\mathbf{x})}_{\mathcal{H}\{p(\mathbf{x})\}} = -Q(\mathbf{x}) \qquad \mathbf{x} \in \Omega \subseteq \mathbb{R}^d$$

1. Testing using the test function $\psi(\mathbf{x})$

$$\int_\Omega \psi(\mathbf{x}) \left[\nabla^2 p(\mathbf{x}) + k^2 p(\mathbf{x})\right] \,\mathrm{d}\mathbf{x} = \int_\Omega -\psi(\mathbf{x}) Q(\mathbf{x}) \,\mathrm{d}\mathbf{x}$$

2. Integration by parts (twice)

$$\psi \nabla^2 p = \nabla \cdot (\psi \nabla p) - \nabla \psi \cdot \nabla p = \nabla \cdot (\psi \nabla p) - \nabla \cdot (\nabla \psi p) + \nabla^2 \psi p$$

Use this in the formula

$$\int_\Omega \psi \nabla^2 p \,\mathrm{d}\mathbf{x} + \int_\Omega \psi k^2 p \,\mathrm{d}\mathbf{x} = \int_\Omega -\psi Q \,\mathrm{d}\mathbf{x}$$

[1] i.e., the right hand side is non-zero

3. Result of integration by parts is (with $\mathrm{d}\mathbf{x}$ not shown)

$$\int_\Omega \nabla\cdot(\psi\nabla p) - \int_\Omega \nabla\cdot(\nabla\psi p) + \int_\Omega \nabla^2\psi p + \int_\Omega \psi k^2 p = \int_\Omega -\psi Q$$

4. Apply Gauss theorem on the first two integrals

$$\int_\Gamma \psi\frac{\partial p}{\partial n}\,\mathrm{d}\mathbf{x} - \int_\Gamma \frac{\partial\psi}{\partial n} p\,\mathrm{d}\mathbf{x} + \int_\Omega \underbrace{\left[\nabla^2\psi + k^2\psi\right]}_{\mathcal{H}\{\psi(\mathbf{x})\}} p\,\mathrm{d}\mathbf{x} = \int_\Omega -\psi Q\,\mathrm{d}\mathbf{x}$$

Notice that the Helmholtz operator $\mathcal{H}$ acts on the test function $\psi(\mathbf{x})$. We exploit this property in the next step.

5. Apply free field Green's function as $\psi(\mathbf{x}) = G(\mathbf{x}, \mathbf{x}_0)$

$$\int_\Gamma G\frac{\partial p}{\partial n}\,\mathrm{d}\mathbf{x} - \int_\Gamma \frac{\partial G}{\partial n} p\,\mathrm{d}\mathbf{x} - \alpha(\mathbf{x}_0)p(\mathbf{x}_0) = \int_\Omega -GQ\,\mathrm{d}\mathbf{x}$$

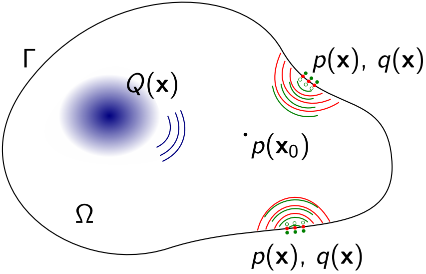
with $\alpha(\mathbf{x}_0) = 1$, $1/2$, or 0 (in Ω, Γ, or otherwise)

Physical interpretation

- Using reciprocity: $G(\mathbf{x}, \mathbf{x}_0) = G(\mathbf{x}_0, \mathbf{x})$ leads to the physical interpretation of the boundary integral representation
- Let $\mathbf{x}_0$ be inside the domain Ω, then

$$p(\mathbf{x}_0) = \underbrace{\int_\Gamma G(\mathbf{x}_0, \mathbf{x}) \frac{\partial p(\mathbf{x})}{\partial n}\,\mathrm{d}\mathbf{x}}_{\text{scattered field (part I)}} - \underbrace{\int_\Gamma \frac{\partial G(\mathbf{x}_0, \mathbf{x})}{\partial n} p(\mathbf{x})\,\mathrm{d}\mathbf{x}}_{\text{scattered field (part II)}} + \underbrace{\int_\Omega G(\mathbf{x}_0, \mathbf{x}) Q(\mathbf{x})\,\mathrm{d}\mathbf{x}}_{\text{incident field}}$$

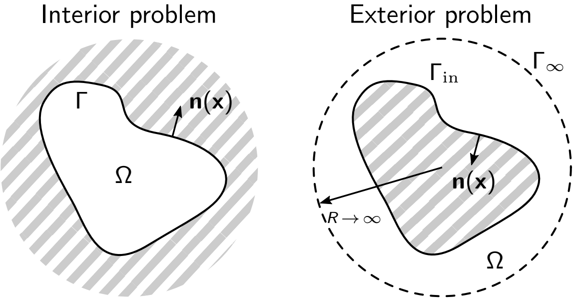
- Denote $\frac{\partial p}{\partial n} = q$ and use the Euler equation: $q = -\mathrm{j}\omega\rho_0 v_n$



- Incident field $p_{\mathrm{inc}}(\mathbf{x}_0)$
- Scattered field $p_{\mathrm{scat}}(\mathbf{x})$
  - Monopole distribution
    Source: surface velocity v_n
  - Dipole distribution
    Source: surface pressure p

$$p(\mathbf{x}_0) = p_{\mathrm{inc}}(\mathbf{x}_0) + p_{\mathrm{scat}}(\mathbf{x})$$

The Sommerfeld radiation condition



- Exterior problem: the boundary is composed of the finite boundary and the infinitely far boundary: $\Gamma = \Gamma_{\mathrm{in}} \cup \Gamma_\infty$
- Sommerfeld's condition:
  - Mathematical statement: the boundary integral on Γ_∞ must vanish in free field conditions, i.e.:

$$\int_{\Gamma_\infty} \left(G \frac{\partial p}{\partial n} - \frac{\partial G}{\partial n} p \right) = 0$$

  - Physical meaning: no energy is reflected back from infinity

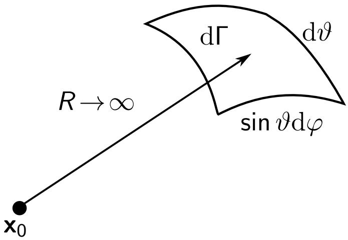
Sommerfeld condition in 3D

- For example, in 3D the integral surely vanishes on Γ_∞

$$\lim_{R\to\infty} \int_\Gamma \left(\frac{\mathrm{e}^{-\mathrm{j}kr}}{4\pi r} \frac{\partial p}{\partial r} - \frac{\partial}{\partial r} \left(\frac{\mathrm{e}^{-\mathrm{j}kr}}{4\pi r} \right) p \right) \mathrm{d}\Gamma = 0$$

- If it vanishes on all small patches $\mathrm{d}\Gamma$
  (Note: G and p are constant on the small patch)

$$\lim_{R\to\infty} R^2 \left(\frac{\mathrm{e}^{-\mathrm{j}kR}}{4\pi R} \frac{\partial p}{\partial r} + (\cancel{1} + \mathrm{j}kR) \frac{\mathrm{e}^{-\mathrm{j}kR}}{4\pi R^2} p \right) \sin\vartheta \mathrm{d}\varphi \mathrm{d}\vartheta = 0$$



- Drop the constants $\sin\vartheta$, $\mathrm{e}^{-\mathrm{j}kR}$, 4π, $\mathrm{d}\vartheta$, $\mathrm{d}\varphi$ to get

$$\lim_{R\to\infty} R\left[\frac{\partial p}{\partial r} + \mathrm{j}kp\right] = 0$$

- Using the Euler equation $\frac{\partial p}{\partial r} = -\mathrm{j}\omega\rho_0 v_r$ we get

$$\lim_{R\to\infty} R\left[p - z_0 v_r\right] = 0$$

Note $z_0 = \rho_0 c$ is the specific plane wave impedance

- Similarly, in d dimensions we have

$$\lim_{R\to\infty} R^{\frac{d-1}{2}}\left[p - z_0 v_r\right] = 0$$

- We can verify that the free field Green's functions all satisfy the Sommerfeld condition ($p = G_1$, G_2, or G_3 above)
- This means that for any radiator radiating finite power the boundary integrals on Γ_∞ can be omitted

Discretization

- We need to solve the KHIE using a numerical method
- Zero incident field is assumed in the following for simplicity
- We discretize the boundary into boundary elements

$$\Gamma \approx \bigcup_{i=1}^{E} \Gamma_i \qquad \text{with} \quad \Gamma_i \cap \Gamma_j = \emptyset \quad \text{if} \quad i \neq j$$

- The general method of data discretization is to approximate the boundary data by a finite number of so-called shape functions $N(\mathbf{x})$

$$p(\mathbf{x}) = \sum_j N_j^{(p)}(\mathbf{x}) p_j \qquad \mathbf{x} \in \Gamma$$

$$q(\mathbf{x}) = \frac{\partial p(\mathbf{x})}{\partial n} = \sum_j N_j^{(q)}(\mathbf{x}) q_j \qquad \mathbf{x} \in \Gamma$$

- In the BEM we have a great freedom in choosing the shape functions. Here, we will consider the simplest choice: the piecewise constant approximation

The collocation form

- Shape and test function choices:
  - $N_j^{(p)}$, $N_j^{(q)}$: piecewise constant over the j-th element
- With this we have E unknowns, we also need E equations
- We choose the E collocational points $\mathbf{x}_i$ $(i = 1 \dots E)$ by setting $\mathbf{x}_i$ to the center of the i-th element

$$\frac{1}{2} p_i = \sum_j \underbrace{\int_{\Gamma_j} G(\mathbf{x}_i, \mathbf{x}) N_j^{(q)}(\mathbf{x}) \, \mathrm{d}\mathbf{x}}_{G_{ij}} \; q_j - \sum_j \underbrace{\int_{\Gamma_j} G_n'(\mathbf{x}_i, \mathbf{x}) N_j^{(p)}(\mathbf{x}) \, \mathrm{d}\mathbf{x}}_{H_{ij}} \; p_j$$

- The shape functions are non-zero only over one element, thus, integration is carried out element-by-element
- Matrix elements by integration of the fundamental solution (and its normal derivative) over one boundary element:

$$G_{ij} = \int_{\Gamma_j} G(\mathbf{x}_i, \mathbf{x}) \, \mathrm{d}\mathbf{x} \qquad (i, j) = 1 \dots E$$

$$H_{ij} = \int_{\Gamma_j} \frac{\partial G(\mathbf{x}_i, \mathbf{x})}{\partial n(\mathbf{x})} \, \mathrm{d}\mathbf{x} \qquad (i, j) = 1 \dots E$$

The BEM system of equations

- Matrix form:

$$\frac{1}{2}\mathbf{p} = \mathbf{G}\mathbf{q} - \mathbf{H}\mathbf{p}$$

  ($\mathbf{p}$ and $\mathbf{q}$ are column vectors of the unknowns)
- From the BCs either $\mathbf{p}$ or $\mathbf{q}$ is known[2]
- Solution for $\mathbf{p}$ (scattered pressure field over the surface)

$$\mathbf{p} = \left(\mathbf{H} + \frac{1}{2}\mathbf{I}\right)^{-1} \mathbf{G}\mathbf{q}$$

- Common properties of matrices $\mathbf{G}$ and $\mathbf{H}$
  1. Fully populated (size $E \times E$)
  2. Complex valued
  3. *Frequency* (wave number k) dependent
  4. Contain singular integrals over elements
  5. Asymmetric (in case of the collocational formalism)

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[2] More precisely: for each element i, either p_i, q_i, or their linear combination must be known

Computing the radiated field

- For the computation of the radiated field, the boundary integral representation (BIR) is used.

$$p(\mathbf{x}_0) = \int_\Gamma G(\mathbf{x}, \mathbf{x}_0) q(\mathbf{x})\,\mathrm{d}\mathbf{x} - \int_\Gamma G'_n(\mathbf{x}, \mathbf{x}_0) p(\mathbf{x})\,\mathrm{d}\mathbf{x} \qquad \mathbf{x}_0 \in \Omega$$

  Notice, that this is a simple forward step, as we already know the surface quantities
- As we have discretized the surface variables, the integrals on the r.h.s. can be written as matrix–vector products.
- If we choose a number of field points, we get the field point pressures $\mathbf{p}_f$ by a simple multiplication

$$\mathbf{p}_f = \mathbf{G}_f \mathbf{q}_s - \mathbf{H}_f \mathbf{p}_s$$

- $\mathbf{G}_f$ and $\mathbf{H}_f$ are also full (size $M \times E$, M: number of field points), frequency dependent, but contain no singular integrals

Acoustical BEM – solution steps

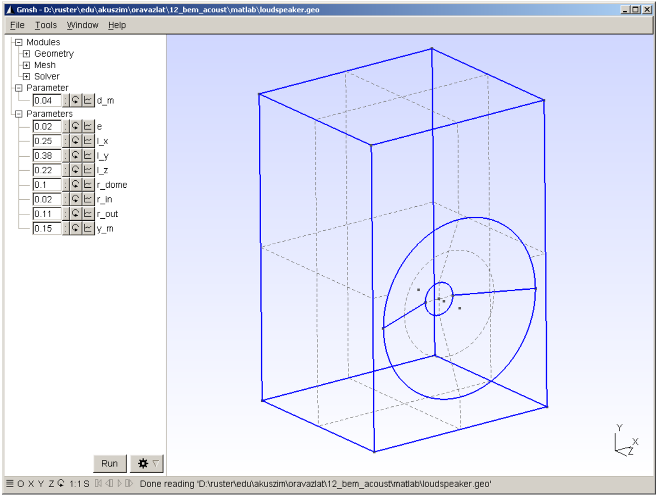
Generally, the solution of an acoustical BEM problem consists of the following steps:

1. Create a discretized geometry (mesh) of the problem
2. Define the boundary conditions (BCs) over the whole surface
3. Set the test (angular) frequency ω
4. Compute the surface matrices $\mathbf{G}$ and $\mathbf{H}$, and the field matrices $\mathbf{G}_f$, $\mathbf{H}_f$ using numerical integration
5. Solve the BEM equation to get the missing surface quantities (solve the full system with matrices $\mathbf{G}$ and $\mathbf{H}$)
6. Use the boundary integral representation formula to calculate the radiated field in the field points (use $\mathbf{G}_f$, $\mathbf{H}_f$)

Multi-frequency analysis: repeat steps 3–6 for each test frequency.
Steps 4 and 5 take the most time and computational effort.

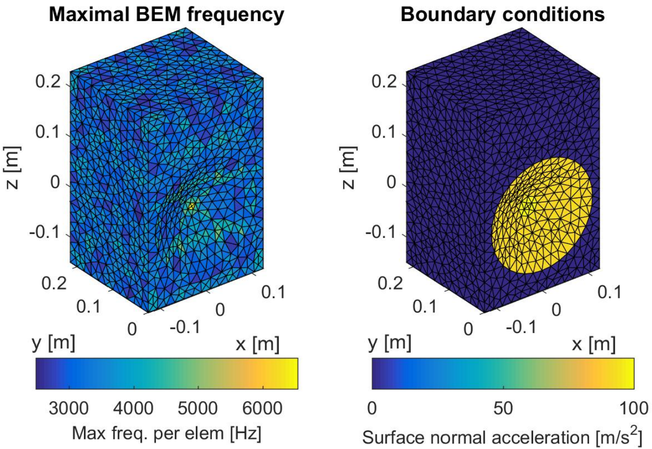
Example – A radiation problem I.

- Example problem: exterior radiation from a loudspeaker
- No sources inside the domain (i.e., zero incident field), sound is generated by a vibrating surface (membrane of the speaker)



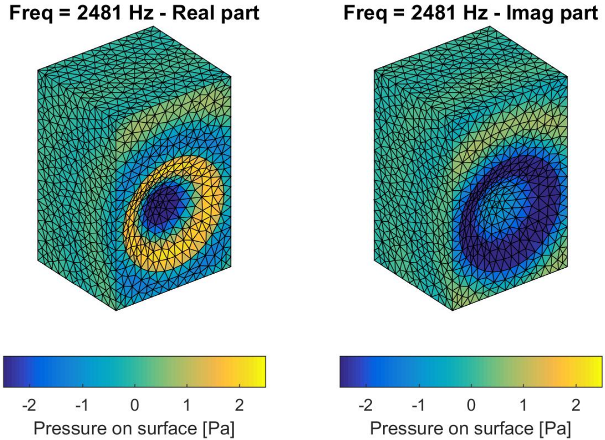
Example – A radiation problem II.

- Computation at different frequencies, maximum frequency is limited by the largest elements (rule of thumb: $l_e < \lambda/6$)
- Constant acceleration on the membrane is assumed
- The membrane is not planar $\rightarrow$ normal velocity is not constant



Example – A radiation problem III.

- Solution process (for each frequency)
  1. Assembe matrices $\mathbf{G}$, $\mathbf{H}$, $\mathbf{G}_f$, $\mathbf{H}_f$
  2. Compute surface pressure by solving the KHIE
  3. Compute field point pressures by using the BIR



Example – A radiation problem IV.

- Field point result – Directivity of the loudspeaker
- Frequency dependency is clearly observed
  - Low frequencies – radiation is nearly spherical
  - Higher frequencies – focused radiation, side lobes appear
  - Vertical directivity is asymmetric, as expected