Kirchhoff-type modelling of concentrated parameter mechanical systems

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Engineering Acoustics Lecture Notes

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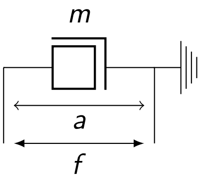


Assumptions:

- acceleration proportional to excitation force

Newton's second law

$$f(t) = ma(t) \tag{1}$$

where

t is time $[\mathrm{s}]$

f is excitation force $[\mathrm{N}]$

a is acceleration $[\mathrm{m/s^2}]$

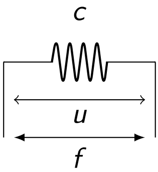
m is mass $[\mathrm{kg}]$

Note: acceleration is measured relative to a *fixed* reference position (infinite mass)

Concentrated stiffness /compliance

Assumptions:

- deformation proportional to excitation force



Hooke's law

$$f(t) = ku(t) = \frac{1}{c}u(t) \tag{2}$$

where

f is excitation force $[\mathrm{N}]$

u is linear deformation $(L(t) - L_0)$ $[\mathrm{m}]$

k is stiffness $[\mathrm{N/m}]$

c is compliance $[\mathrm{m/N}]$

Concentrated stiffness /compliance

Hooke's law in more general form:

$$\sigma(t) = E\epsilon(t) \tag{3}$$

where

- σ is mechanical stress (force per unit area) $[\mathrm{N/m^2}]$
- ϵ is strain $(\frac{L-L_0}{L_0})$ $[-]$
- E is Young's modulus of elasticity $[\mathrm{N/m^2}]$ – material parameter

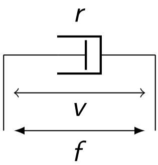
Relation of stiffness / compliance to material and geometrical parameters:

$$k = \frac{EA}{L_0}, \quad c = \frac{L_0}{EA} \tag{4}$$

Concentrated viscous damping

Assumptions:

- velocity proportional to excitation force



$$f(t) = rv(t) \tag{5}$$

where

f is excitation force $[\mathrm{N}]$

v is velocity of deformation $\frac{\mathrm{d}}{\mathrm{d}t}(L(t) - L_0)$ $[\mathrm{m/s}]$

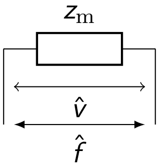
r is viscous damping $[\mathrm{Ns/m}]$

Mechanical impedance

Impedance is interpreted in frequency domain: $f(t) = \hat{f}(\omega)\mathrm{e}^{\mathrm{j}\omega t}$

Assumptions:

- velocity proportional to excitation force



$$\hat{f}(\omega) = z_\mathrm{m}(\omega)\hat{v}(\omega) \tag{6}$$

where

$\hat{f}$ is complex amplitude of excitation force $[\mathrm{N}]$

$\hat{v}$ is complex amplitude of velocity of deformation $[\mathrm{m/s}]$

z_m is mechanical impedance $[\mathrm{Ns/m}]$

Mechanical impedance of concentrated elements

- mechanical impedance of mass:

$$z_{\mathrm{mass}} = \frac{\hat{f}}{\hat{v}} = \frac{\hat{f}}{\frac{\hat{a}}{\mathrm{j}\omega}} = \mathrm{j}\omega m \tag{7}$$

- mechanical impedance of compliance:

$$z_{\mathrm{comp}} = \frac{\hat{f}}{\hat{v}} = \frac{\hat{f}}{\mathrm{j}\omega\hat{u}} = \frac{1}{\mathrm{j}\omega c} \tag{8}$$

- mechanical impedance of damping:

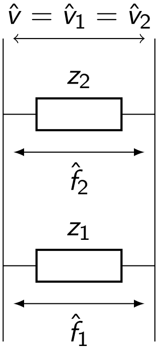
$$z_{\mathrm{damp}} = \frac{\hat{f}}{\hat{v}} = r \tag{9}$$

Dynamic mechanical transfer quantities

$\hat{f}/\hat{u}$: dynamic stiffness	$\hat{u}/\hat{f}$: dynamic compliance
$\hat{f}/\hat{v}$: impedance	$\hat{v}/\hat{f}$: admittance
$\hat{f}/\hat{a}$: dynamic mass	$\hat{a}/\hat{f}$: dynamic mobility

Connecting mechanical impedances

Common velocity

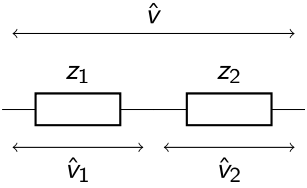


$$\hat{v} = \hat{v}_1 = \hat{v}_2, \qquad (10)$$

$$\hat{f} = \hat{f}_1 + \hat{f}_2, \qquad (11)$$

$$z = z_1 + z_2 \qquad (12)$$

Common force

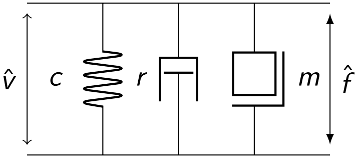


$$\hat{f} = \hat{f}_1 = \hat{f}_2, \qquad (13)$$

$$\hat{v} = \hat{v}_1 + \hat{v}_2, \qquad (14)$$

$$z = z_1 \times z_2 \qquad (15)$$

SDOF damped oscillator



$$\hat{v}(s) = \hat{f}(s) \cdot \frac{1}{z} = \hat{f}(s) \cdot \frac{1}{sm + r + \frac{1}{sc}} = \hat{f}(s) \cdot \frac{sc}{1 + \frac{1}{Q}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2} \tag{16}$$

with natural frequeny ω_0 and quality factor Q written as

$$\omega_0 = \frac{1}{\sqrt{mc}}, \quad Q = \frac{\sqrt{m}}{r\sqrt{c}} \tag{17}$$

or equivalently

$$\hat{u}(s) = \frac{\hat{v}(s)}{s} = \hat{f}(s)c \cdot \frac{1}{1 + \frac{1}{Q}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2} \tag{18}$$

SDOF damped oscillator

$$\frac{\hat{u}(s)}{\hat{f}(s) \cdot c} = \frac{1}{1 + \frac{1}{Q}\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2} = \frac{1}{1 + \frac{1}{Q}\frac{\mathrm{j}\omega}{\omega_0} - \left(\frac{\omega}{\omega_0}\right)^2} \tag{19}$$

- low frequency: $\omega < \omega_0$
  Transfer ≈ 1
- mid frequency: $\omega \approx \omega_0$
  Transfer determined by quality factor Q (damping)
  Amplification at $\omega = \omega_0$: Q
- high frequency: $\omega > \omega_0$
  Transfer $\approx 1/\omega^2$
  Asymptotically: $-12\ \mathrm{dB}$ per octave (-40 dB per decade)

SDOF damped oscillator

Response to Dirac delta force excitation $\hat{f}(s) = 1$ (with damping factor $\xi = 1/2Q$):

$$\hat{u}(s) = c \cdot \frac{1}{1 + \frac{1}{Q}\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2} = c \cdot \frac{1}{1 + 2\xi\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2} \tag{20}$$

poles of transfer function

$$s = -\xi\omega_0 \pm \mathrm{j}\omega_0\sqrt{1-\xi^2} \tag{21}$$

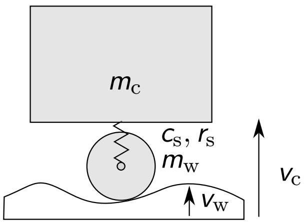
Time domain solution (with inverse Laplace transform):

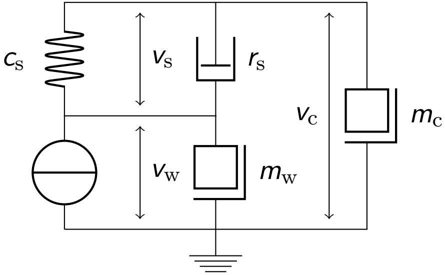
$$u(t) = \frac{c\omega_0}{\sqrt{1-\xi^2}}\mathrm{e}^{-t/\tau}\sin(\omega_\mathrm{d} t) \tag{22}$$

with time constant and damped natural frequency written as

$$\tau = \frac{1}{\omega_0\xi}, \quad \omega_d = \omega_0\sqrt{1-\xi^2} \tag{23}$$

Vehicle with suspension





v_w velocity of wheel

v_c velocity of car

m_w mass of wheel

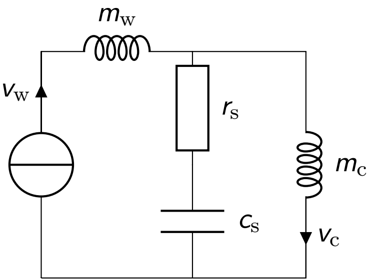
c_s, r_s compliance and damping of suspension

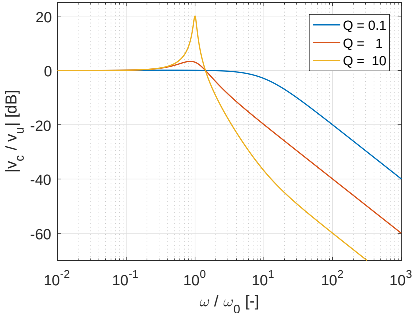
m_c mass of car

Mechano-electrical analogies

electrical	mechanical
voltage	force
current	velocity
inductance	mass
resistance	damping
capacitance	compliance
impedance	impedance
parallel	common force
serial	common velocity

Vehicle with suspension





$$\frac{v_c}{v_w} = \frac{r_s + \frac{1}{sc_s}}{r_s + \frac{1}{sc_s} + sm_c} = \frac{1 + \frac{1}{Q}\frac{s}{\omega_0}}{1 + \frac{1}{Q}\frac{s}{\omega_0} + \left(\frac{s}{\omega_0}\right)^2} \tag{24}$$

$$\omega_0 = \frac{1}{\sqrt{m_c c_s}}, \qquad Q = \frac{\sqrt{m_c}}{r_s\sqrt{c_s}} \tag{25}$$

A dynamic interaction problem

- velocity generator v_{g} with finite impedance z_{g} loaded by impedance z_{l}

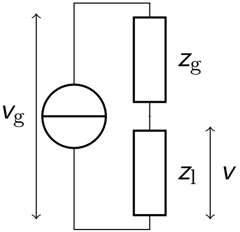
State I – disconnected

- generator vibrates with velocity v_{g}
- structure stands still $v = 0$

State II – connected

- both components vibrate with velocity v
- velocities modified by common interaction (contact) force f_{c}

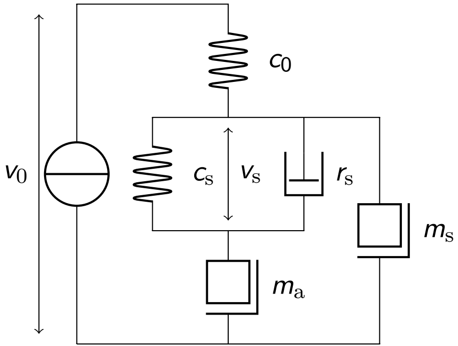
$$f_{\mathrm{c}} = z_{\mathrm{g}} \cdot (v_{\mathrm{g}} - v) = z_{\mathrm{l}} \cdot v \qquad (26)$$



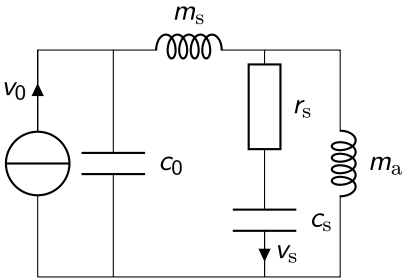
$$f_{\mathrm{c}} = v_{\mathrm{g}} \left(z_{\mathrm{g}} \times z_{\mathrm{l}} \right), \quad v = \frac{f_{\mathrm{c}}}{z_{\mathrm{l}}} = \frac{z_{\mathrm{g}}}{z_{\mathrm{g}} + z_{\mathrm{l}}} \qquad (27)$$

Vinyl pickup

mechanical circuit



mechano-electrical analog



Vinyl pickup

Transfer at low frequencies: $1/\omega c_0 \gg \omega(m_s + m_a)$

$$\frac{v_\mathrm{s}}{v_0} = \frac{s m_\mathrm{a}}{s m_\mathrm{a} + r_\mathrm{s} + \frac{1}{s c_\mathrm{s}}} = \frac{\left(\frac{s}{\omega_1}\right)^2}{1 + \frac{1}{Q_1}\left(\frac{s}{\omega_1}\right) + \left(\frac{s}{\omega_1}\right)^2} \qquad (28)$$

where

$$\omega_1 = 1/\sqrt{m_\mathrm{a} c_\mathrm{s}}, \qquad Q_1 = \frac{\sqrt{m_\mathrm{a}}}{r_\mathrm{s}\sqrt{c_\mathrm{s}}} \qquad (29)$$

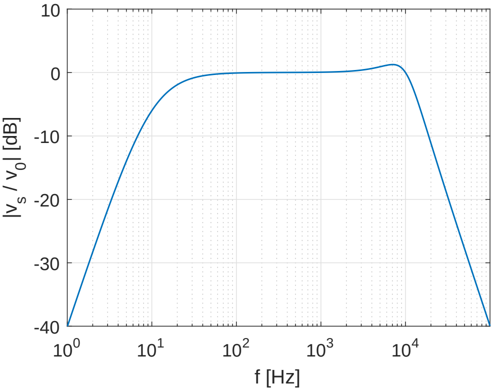
Transfer at high frequencies: $1/\omega c_\mathrm{s} \ll r_\mathrm{s}, \omega m_\mathrm{a} \gg r_\mathrm{s}$

$$\frac{v_\mathrm{s}}{v_0} = \frac{\frac{1}{s c_0}}{\frac{1}{s c_0} + s m_\mathrm{s} + r_\mathrm{s}} = \frac{1}{1 + \frac{1}{Q_2}\left(\frac{s}{\omega_2}\right) + \left(\frac{s}{\omega_2}\right)^2} \qquad (30)$$

where

$$\omega_2 = 1/\sqrt{m_\mathrm{s} c_0}, \qquad Q_2 = \frac{\sqrt{m_\mathrm{s}}}{r_\mathrm{s}\sqrt{c_0}} \qquad (31)$$

Vinyl pickup



Parameters:

- $m_\mathrm{s} = 0.1\ \mathrm{g}$
- $\omega_1 = 2\pi \cdot 10\ \mathrm{Hz},\ Q_1 = 0.5$
- $m_\mathrm{a} = 50\ \mathrm{g}$
- $\omega_2 = 2\pi \cdot 10\ \mathrm{kHz},\ Q_2 = 1$

Tuned mass dampers

- Mass-spring oscillator (M, C) excited by wideband force $\hat{f}(s)$
- Velocity response $\hat{v}(s)$ in frequency domain

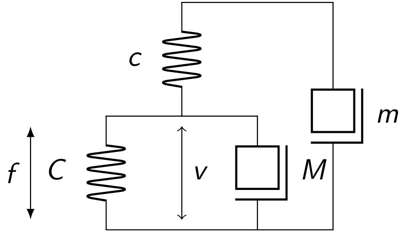
$$\hat{v}(s) = \hat{f}(s) \cdot \frac{sC}{1 + \left(\frac{s}{\omega_0}\right)^2} \tag{32}$$

  where $\omega_0 = \frac{1}{\sqrt{MC}}$

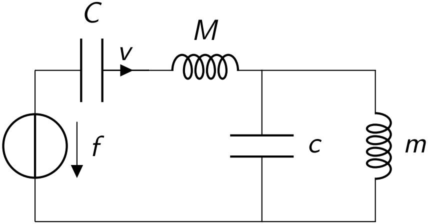
- Tuned mass damper: extend the system by an other mass-spring oscillator (m, c) tuned to the same frequency $\omega_0 = \frac{1}{\sqrt{mc}}$

Tuned mass dampers

mechanical circuit



mechano-electrical analog



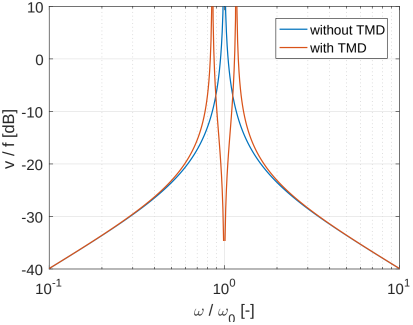
input impedance without TMD ($m = 0$)

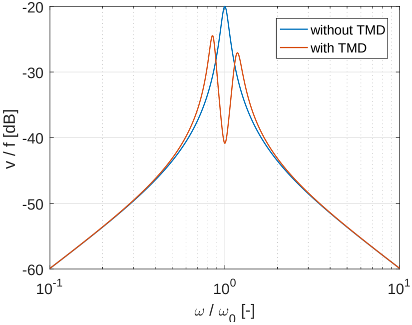
$$z = \frac{1}{sC} + sM = \frac{1 + s^2 MC}{sC} \tag{33}$$

input impedance with TMD

$$z = \frac{1}{sC} + sM + \frac{1}{sc} \times sm = \underbrace{\frac{1 + s^2 MC}{sC}}_{0} + \underbrace{\frac{sm}{1 + s^2 mc}}_{\infty} \tag{34}$$

Tuned mass dampers





- $M/m = 10$
- $Q = q = \infty$

- $M/m = 10$
- $Q = q = 10$