The 3-dimensional sound field

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Engineering Acoustics Lecture Notes

Wave equation in 3D

Derivation of Euler's equation
Newton's second law for the sound particle

$$\int_A -P(\mathbf{x}, t)\mathbf{n}(\mathbf{x})\mathrm{d}A(\mathbf{x}) = \int_V \rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x}, t)\mathrm{d}V(\mathbf{x}) \qquad (1)$$

Application of Gauss' theorem to the surface integral:

$$\int_V -\nabla P(\mathbf{x}, t)\mathrm{d}V(\mathbf{x}) = \int_V \rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x}, t)\mathrm{d}V(\mathbf{x}) \qquad (2)$$

Limiting to $V \to 0$ yields Euler's equation

$$\nabla P(\mathbf{x}, t) + \rho(\mathbf{x})\ddot{\mathbf{u}}(\mathbf{x}, t) = 0 \qquad (3)$$

Wave equation in 3D

Gas law for adiabatic state changes, linearized around V_0, P_0:

$$p = -\kappa P_0 \frac{\mathrm{d}V}{V_0} \tag{4}$$

Application for the sound particle:

$$\begin{aligned} p(\mathbf{x}, t) &= -\kappa P_0 \frac{\int_{A_0} \mathbf{u}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}) \mathrm{d}A(\mathbf{x})}{V_0} \\ &= -\kappa P_0 \frac{\int_{V_0} \nabla \cdot \mathbf{u}(\mathbf{x}, t) \mathrm{d}V(\mathbf{x})}{V_0} \\ &\rightarrow -\kappa P_0 \nabla \cdot \mathbf{u}(\mathbf{x}, t) \end{aligned} \tag{5}$$

Wave equation in 3D

Combination of (3) and (5):

$$\nabla \cdot \nabla p(\mathbf{x}, t) + \frac{1}{c^2}\ddot{p}(\mathbf{x}, t) = 0 \tag{6}$$

where $\nabla \cdot \nabla$ is $\mathrm{divgrad} = \nabla^2 = \Delta$ is the Laplace operator, and

$$c = \sqrt{\frac{\kappa P_0}{\rho_0}} \tag{7}$$

is speed of sound.
In frequency domain $\rightarrow$ Helmholtz equation

$$\nabla^2 p(\mathbf{x}, \omega) + k^2 p(\mathbf{x}, \omega) = 0 \tag{8}$$

with $k = \omega / c$

Solution with spherical symmetry

Spherical coordinates:

$$\mathbf{x} = (r, \theta, \phi), \quad \mathbf{u}(\mathbf{x}) = (u_r(\mathbf{x}), u_\theta(\mathbf{x}), u_\phi(\mathbf{x})) \tag{9}$$

Spherical symmetry: quantities depend only on the distance from the origin r

$$f(\mathbf{x}) = f(r) \tag{10}$$

Laplace operator with spherical symmetry:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} \tag{11}$$

Gradient with spherical symmetry:

$$\nabla = \begin{bmatrix} \partial/\partial r \\ 0 \\ 0 \end{bmatrix} \tag{12}$$

Solution of the 3D Helmholtz equation

Assumption: $p(r) = \frac{\Psi(r)}{r}$
Substitution into the Helmholtz equation and application of the Laplace operator with spherical symmetry yields the 1D Helmholtz equation for the numerator $\Psi(r)$

$$\Psi''(r) + k^2\Psi(r) = 0 \tag{13}$$

leading to the solution

$$p(r) = p^+ \frac{\mathrm{e}^{-\mathrm{j}kr}}{r} + p^- \frac{\mathrm{e}^{+\mathrm{j}kr}}{r} \tag{14}$$

Physical interpretation:

- p^+: outgoing waves with magnitude decay $1/r$.
- p^-: incoming (focusing) waves with magnitude increase $1/r$

Spherical specific impedance

Assumption: only outgoing waves: $p^- = 0$ (Sommerfeld radiation condition)
Relation of pressure to particle velocity

$$z_s(r) = \frac{p(r)}{v(r)} \tag{15}$$

Velocity is determined from Euler's equation

$$v(r) = \frac{-1}{\mathrm{j}\omega\rho} p'(r) = \frac{1}{z_0} \frac{\mathrm{e}^{-\mathrm{j}kr}}{r} \frac{(1 + \mathrm{j}kr)}{\mathrm{j}kr} \tag{16}$$

resulting in

$$z_s(r) = z_0(1 \times \mathrm{j}kr) \tag{17}$$

Pulsating sphere

Sphere of radius R pulsating with velocity $v(R) = v_0$
Pressure on sphere surface:

$$p_0 = p(R) = v_0 z_s(R) = v_0 z_0 (1 \times \mathrm{j}kR) \tag{18}$$

radiated pressure:

$$p(r) = p_0 \frac{\mathrm{e}^{-\mathrm{j}k(r-R)}}{r/R} = v_0 z_0 (1 \times \mathrm{j}kR) \frac{\mathrm{e}^{-\mathrm{j}k(r-R)}}{r/R} \tag{19}$$

Acoustic monopole

Acoustic monopole: pulsating sphere with radius $R \to 0$, but velocity $v \to \infty$ so that surface volume velocity $Q = v_0 A = v_0 4R^2\pi$ remains constant
Field of monopole:
Substituting $v_0 = \frac{Q}{4R^2\pi}$ into the field of the pulsating sphere and taking the limit $R \to 0$ yields

$$p(r) = \mathrm{j}\omega Q\rho \frac{\mathrm{e}^{-\mathrm{j}kr}}{4\pi r} \tag{20}$$

Sound power of acoustic monopole

Due to spherical symmetry, Sound power S is computed as

$$S = \int_A \overline{I} \mathrm{d}A = \overline{I} \cdot 4\pi r^2 \tag{21}$$

The mean intensity is computed by definition as

$$\overline{I} = \frac{1}{2} \Re \left(p v^* \right) \tag{22}$$

where p is given from (20), and the velocity is computed as

$$v(r) = p(r)/z_s(r), \quad z_s(r) = z_0(1 \times \mathrm{j}kr) \tag{23}$$

Summarizing:

$$S = \frac{Q^2 k^2 z_0}{8\pi} \tag{24}$$

Acoustic dipole

Two monopoles at distance Δz, with opposite source strengths $\pm Q$
Field of dipole:

$$p_d(x,y,z) = p_m(x,y,z-\Delta z/2) - p_m(x,y,z+\Delta z/2) \tag{25}$$

$$\rightarrow_{\Delta z \rightarrow 0} -\frac{\partial p_m(x,y,z)}{\partial z}\Delta z \tag{26}$$

$$= \frac{\mathrm{j}\omega\rho\mu}{4\pi}\frac{\mathrm{e}^{-\mathrm{j}kr}}{r}\frac{1+\mathrm{j}kr}{r}\cos\theta \tag{27}$$

where $\mu = Q\Delta z$ is the monopole strength

Sound power of the acoustic dipole

Sound power of dipole (approximation)

$$S_d = \frac{\mu^2 k^4 z_0}{12\pi} \tag{28}$$

Physical interpretation:

$$S_m \propto k^2 \qquad S_d \propto k^4 \tag{29}$$

Difference is interpreted as „acoustic short circuit" Low frequency pressure fluctuations around the dipole are shunted by local wave motion around the dipole, and no energy is radiated in the far field.

Field of an acoustic line source

Line source interpretatation: continuous distribution of monopole sources along the z axis

Infinitesimal source strength: $\mathrm{d}Q = v(z)\mathrm{d}z 2\pi a$, where a is the line source's radius

Far field of line source

$$p(x,z) = \int_{-\infty}^{\infty} \frac{\mathrm{j}\omega\rho 2\pi a v(z)\mathrm{d}z \mathrm{e}^{-\mathrm{j}k(r - z\sin\theta)}}{4\pi(r - z\sin\theta)} \quad (30)$$

$$\approx \frac{\mathrm{j}\omega\rho 2\pi a \mathrm{e}^{-\mathrm{j}kr}}{4\pi r} \int_{-\infty}^{\infty} v(z)\mathrm{e}^{\mathrm{j}kz\sin\theta}\mathrm{d}z \quad (31)$$

$$= \mathrm{j}\omega\rho 2\pi a \frac{\mathrm{e}^{-\mathrm{j}kr}}{4\pi r} V(k_z) \quad (32)$$

where $V(k_z)$ is the wave number spectrum (spatial Fourier transform) of the velocity excitation $v(z)$

Directivity and wave number spectrum

Directivity is closely related to wave number spectrum.
For the case of the line source, its directivity in direction θ can be read from the wave number spectrum $V(k_z)$ with the substitution $k_z = k \sin\theta$
Directivity of a finite homogeneous line source:

$$V(k_z) = v_0 \int_{-L/2}^{L/2} \mathrm{e}^{\mathrm{j}k_z z} \mathrm{d}z = v_0 L \mathrm{sinc}\left(k_z L/2\right) \tag{33}$$

resulting in the far field

$$p(x, z) = \mathrm{j}\omega\rho Q \frac{\mathrm{e}^{-\mathrm{j}kr}}{4\pi r} \mathrm{sinc}\left(k_z L/2\right) \tag{34}$$

The Rayleigh integral

Sound radiation from an infinite rigid plane
Rigid plane assumption: Monopole radiator elements
pressure field

$$p(\mathbf{x}) = 2\mathrm{j}\omega\rho \int_A v(\mathbf{x}_0) \frac{\mathrm{e}^{-\mathrm{j}kr}}{4\pi r} \mathrm{d}A(\mathbf{x}_0) \tag{35}$$

where $r = |\mathbf{x} - \mathbf{x}_0|$

Radiation impedance of a cylindrical piston

Pressure at the piston center from Rayleigh integral

$$p(0) = 2\mathrm{j}\omega\rho \int_A v_0 \frac{\mathrm{e}^{-\mathrm{j}kr}}{4\pi r} \mathrm{d}A \tag{36}$$

$$= 2\mathrm{j}\omega\rho \int_{\phi=0}^{2\pi} \int_{r=0}^{R} v_0 \frac{\mathrm{e}^{-\mathrm{j}kr}}{4\pi r} r \mathrm{d}r \mathrm{d}\phi \tag{37}$$

$$= v_0 z_0 \left(\mathrm{e}^{-\mathrm{j}kR} - 1\right) \tag{38}$$

Far field directivity of a cylindrical piston